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Structure of Solutions of Variational Problems

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Alexander J. Zaslavski

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Alexander J. Zaslavski
Department of Mathematics
The Technion-Israel Institute of Technology
Haifa, Israel

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Preface

This book is devoted to the presentation of progress made during the last 10 years in the studies of the structure of approximate solutions of variational problems considered on subintervals of a real line. We present the results on properties of approximate solutions which are independent of the length of the interval, for all sufficiently large intervals. The results in this book deal with the so-called turnpike property of the variational problems. To have this property means, roughly speaking, that the approximate solutions of the problems are determined mainly by the integrand (objective function) and are essentially independent of the choice of interval and endpoint conditions, except in regions close to the endpoints. Turnpike properties are well known in mathematical economics. The term was first coined by P. Samuelson in 1948 when he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). Now it is well known that the turnpike property is a general phenomenon which holds for large classes of variational problems. For these classes of problems using the Baire category approach, it was shown that the turnpike property holds for a generic (typical) problem. In this book we are interested in individual (non-generic) turnpike results and in sufficient and necessary conditions for the turnpike phenomenon.

Haifa, Israel — Alexander J. Zaslavski

Contents

1

Introduction

In this chapter we introduce and discuss turnpike properties and describe the structure of the book.

The study of optimal control problems and variational problems defined on infinite intervals and on sufficiently large intervals has been a rapidly growing area of research [3, 4, 7, 9, 10, 11, 12, 13, 16, 18, 19, 20, 23, 25, 30, 31, 32, 33, 39, 40, 51]. These problems arise in engineering [1, 21, 53], in models of economic growth [2, 15, 17, 24, 29, 34, 35, 37, 51], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [6, 38], and in the theory of thermodynamical equilibrium for materials [14, 22, 26, 27, 28].

Our book consists of four chapters. Here, in Chap. 1, we describe its structure. In Chap. 2 we study the structure of approximate solutions of nonautonomous variational problems with continuous integrands $f : [0,\infty) \times R^n \times R^n \to R^1$, where R^n is the n-dimensional Euclidean space. In Chap. 3, we study turnpike properties for autonomous variational problems with integrands $f : R^n \times R^n \to R^1$. Finally, in Chap. 4 we study the structure of approximate solutions of variational problems with integrands $f : R^n \times R^n \to R^1$ which are convex functions on $R^n \times R^n$.

Now we describe the structure of the book. We begin in Chap. 2 with the study of the structure of solutions of the variational problems:

$$\int_{T_1}^{T_2} f(t, z(t), z'(t))dt \to \min,\ z(T_1) = x,\ z(T_2) = y, \tag{P}$$

$$z : [T_1, T_2] \to R^n \text{ is an absolutely continuous function,}$$

where $T_1 \geq 0$, $T_2 > T_1$, $x, y \in R^n$, and $f : [0,\infty) \times R^n \times R^n \to R^1$ belongs to a complete metric space of integrands $\mathcal{M}$ which is introduced in Sect. 2.1. Note that these integrands do not satisfy any convexity assumption which is usually used in the calculus of variations.

It is well known that the solutions of the problems (P) exist for integrands f which satisfy two fundamental hypotheses concerning the behavior of the

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integrand as a function of the last argument (derivative): one that the integrand should grow superlinearly at infinity and the other that it should be convex [8, 36]. Moreover, certain convexity assumptions are also necessary for properties of lower semicontinuity of integral functionals which are crucial in most of the existence proofs. For integrands f which do not satisfy the convexity assumption the existence of solutions of the problems (P) is not guaranteed, and in this situation we consider δ-*approximate solutions.*

Let $T_1 \geq 0$, $T_2 > T_1$, $x, y \in R^n$, and $f : [0, \infty) \times R^n \times R^n \to R^1$ be an integrand, and let δ be a positive number. We say that an absolutely continuous (a.c.) function $u : [T_1, T_2] \to R^n$ satisfying $u(T_1) = x$, $u(T_2) = y$ is a δ-approximate solution of the problem (P) if

$$\int_{T_1}^{T_2} f(t, u(t), u'(t))dt \leq \int_{T_1}^{T_2} f(t, z(t), z'(t))dt + \delta$$

for each a.c. function $z : [T_1, T_2] \to R^n$ satisfying $z(T_1) = x, \ z(T_2) = y$.

In Chap. 1 we deal with the so-called turnpike property of the variational problems (P) associated with an integrand f. To have this property means that there exists a bounded continuous function $X_f : [0, \infty) \to R^n$ depending only on f such that for each pair of positive numbers $K, \epsilon > 0$, there exist positive constants $L = L(K, \epsilon)$ and $\delta = \delta(K, \epsilon)$ depending on ϵ, K such that if $u : [T_1, T_2] \to R^n$ is an δ-approximate solution of the problem (P) with

$$T_2 - T_1 \geq L, \ |u(T_i)| \leq K, \ i = 1, 2,$$

then

$$|u(t) - X_f(t)| \leq \epsilon \text{ for all } t \in [T_1 + \tau_1, T_2 - \tau_2],$$

where $\tau_1, \tau_2 \in [0, L]$.

(The precise description of the turnpike property is given in Sect. 2.1.)

If the integrand f possesses the turnpike property, then the solutions of variational problems with f are essentially independent of the choice of time interval and values at the endpoints except in regions close to the endpoints of the time interval. If a point t does not belong to these regions, then the value of a solution at t is closed to a trajectory X_f ("turnpike") which is defined on the infinite time interval and depends only on f. This phenomenon has the following interpretation. If one wish to reach a point A from a point B by a car in an optimal way, then one should turn to a turnpike, spend most of time on it, and then leave the turnpike to reach the required point.

Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [35]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). This property was further investigated for optimal trajectories of models of economic dynamics (see, e.g., [2, 15, 17, 24, 29, 34, 37, 51]). Many turnpike results are collected in [51].

In the classical turnpike theory the function f does not depend on the variable t, is strictly convex on the space $R^n \times R^n$, and satisfies a growth condition common in the literature. In this case, the turnpike property can be established, the turnpike X_f is a constant function, and its value is a unique solution of the maximization problem $f(x, 0) \to \min$, $x \in R^n$.

It was shown in our research, which was summarized in [51], that the turnpike property is a general phenomenon which holds for large classes of variational and optimal control problems without convexity assumptions. For these classes of problems a turnpike is not necessarily a constant function (singleton) but may instead be an nonstationary trajectory (in the discrete time nonautonomous case) [51] or an absolutely continuous function on the interval $[0, \infty)$ as it was described above (in the continuous time nonautonomous case) [44, 45, 51] or a compact subset of the space X (in the autonomous case) [39, 40, 41, 43, 50, 51].

More precisely, in Chap. 2 of [51] we study the turnpike properties for variational problems with integrands which belong to a subspace $\mathcal{M}_{co}$ of $\mathcal{M}$. The subspace $\mathcal{M}_{co} \subset \mathcal{M}$ consists of integrands $f \in \mathcal{M}$ such that the function $f(t, x, \cdot) : R^n \to R^1$ is convex for any $(t, x) \in [0, \infty) \times R^n$. In Chap. 2 of [51] we showed that the turnpike property holds for a generic integrand $f \in \mathcal{M}_{co}$. Namely, we established the existence of a set $\mathcal{F}_{co} \subset \mathcal{M}_{co}$ which is a countable intersection of open everywhere dense sets in $\mathcal{M}_{co}$ such that each $f \in \mathcal{F}_{co}$ has the turnpike property.

In [45] we extend this turnpike result of [51] established for the space $\mathcal{M}_{co}$ to the space of integrands $\mathcal{M}$. We show the existence of a set $\mathcal{F} \subset \mathcal{M}$ which is a countable intersection of open everywhere dense sets in $\mathcal{M}$ such that each $f \in \mathcal{F}$ has the turnpike property. In [45] we show that an integrand $f \in \mathcal{F}$ has a turnpike X_f which is a bounded continuous function.

Therefore according to the results of [45, 51], the turnpike property is a general phenomenon which holds for large classes of variational problems. For these classes of problems, using the Baire category approach, it was shown that the turnpike property holds for a generic (typical) problem. In this book we are interested in individual (nongeneric) turnpike results and in sufficient and necessary conditions for the turnpike phenomenon.

In Chap. 2 we consider the following question. Assume that $f \in \mathcal{M}$ and $X : [0, \infty) \to R^n$ is a bounded continuous function. How to verify if the integrand f possesses the turnpike property and X is its turnpike? In Sect. 2.1 we introduce two properties (P1) and (P2) and show that f has the turnpike property if and only if f possesses the properties (P1) and (P2). The property (P2) means that all approximate solutions of the corresponding infinite horizon variational problem have the same asymptotic behavior while the property (P1) means that if an a.c. function $v : [0, T] \to R^n$ is an approximate solution and T is large enough, then there is $\tau \in [0, T]$ such that $v(\tau)$ is close to $X(\tau)$. This result, which was obtained in [47], is stated in Sect. 2.1 while it is proved in Sects. 2.2–2.6.

In Chap. 2 we also consider the strong turnpike property, which was introduced and studied in [49], for an integrand $f \in \mathcal{M}$ such that the function $f(t,x,\cdot) : R^n \to R^1$ is convex for any $(t,x) \in [0,\infty) \times R^n$. We say that the integrand f possesses the strong turnpike property if the turnpike property holds for f with the function X_f being the turnpike, and moreover, in the definition of the turnpike property, $\tau_1 = 0$ if

$$|v(T_1) - X_f(T_1)| \leq \delta$$

and $\tau_2 = 0$ if

$$|v(T_2) - X_f(T_2)| \leq \delta.$$

This additional condition means that if at the point T_1 (T_2, respectively) the value of the approximate solution v is closed to a trajectory X_f, then the value of the solution v at t is closed to a trajectory X_f for all t except in a region close to the endpoint T_2 (T_1, respectively).

In Sect. 2.1 we state Theorem 2.4 (the second main result of Chap. 2) which was obtained in [49]. According to this result, the integrand f possesses the strong turnpike property with the function X_f being the turnpike if and only if the properties (P1) and (P2) hold and the function X_f is a unique solution of the corresponding infinite horizon variational problem associated with the integrand f.

In Chap. 3 we study turnpike properties for autonomous variational problems with continuous integrands $f : R^n \times R^n \to R^1$ which belong to the subspace of functions $\mathcal{A} \subset \mathcal{M}$ introduced in Sect. 3.1. It should be mentioned that $\mathcal{A}$ is a large space of autonomous integrands which was considered in Chaps. 3–5 of [51]. In [41] and in Chap. 3 of [51] we study the structure of approximate solutions of variational problems (P) with integrands $f \in \mathcal{A}$ and show the existence of a subset $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{A}$ such that each integrand $f \in \mathcal{F}$ has a turnpike property, where the turnpike is a nonempty compact subset of the n-dimensional Euclidean space R^n. More precisely, we show there (using the Baire category approach) that for a generic (typical) integrand $f \in \mathcal{A}$, there exists a nonempty compact set $H(f) \subset R^n$ such that the following property holds:

For each $\epsilon > 0$ there exists a constant $L > 0$ such that if v is a solution of problem (P), then for most of $t \in [T_1, T_2]$ the set $v([t, t+L])$ is equal to $H(f)$ up to ϵ with respect to the Hausdorff metric.

Note that for a generic integrand $f \in \mathcal{A}$ the turnpike $H(f)$ is a nonempty compact subset of R^n which is not necessarily a singleton. It should be mentioned that there exists $f \in \mathcal{A}$ which does not have this turnpike property. In Chap. 3 we prove individual (nongeneric) turnpike results for autonomous variational problems with integrands $f \in \mathcal{A}$.

Let $f \in \mathcal{A}$. A locally absolutely continuous (a.c.) function $v : R^1 \to R^n$ is called (f)-minimal if

$$\sup\{|v(t)| : t \in R^1\} < \infty$$

and if for each $T_1 \in R^1$, $T_2 > T_1$,

$$I^f(T_1, T_2, v) \le I^f(T_1, T_2, u)$$

for any a.c. function $u : [T_1, T_2] \to R^n$ satisfying $u(T_i) = v(T_i)$ for $i = 1, 2$.

We denote by $\mathcal{M}(f)$ the set of all (f)-minimal functions $v : R^1 \to R^n$ and set

$$\mathcal{D}(f) = \cup\{v(R^1) : \ v \in \mathcal{M}(f)\}.$$

We show (see Theorem 3.1) that the set $\mathcal{M}(f)$ is nonempty, the set $\mathcal{D}(f)$ is bounded and closed, and (see Theorems 3.2 and 3.3) approximate solutions of the problem (P) spend most of time in a neighborhood of $\mathcal{D}(f)$. Theorems 3.1–3.3, which were obtained in [48], are stated in Sect. 3.1 and proved in Sects. 3.2–3.5.

In Sects. 3.6–3.10 we continue to study turnpike properties for autonomous variational problems with smooth integrands $f : R^n \times R^n \to R^1$ satisfying some growth conditions. For these integrands a turnpike property was studied in [43] and in Chap. 5 of [51]. In these works we say that an integrand f has the turnpike property if for any $\epsilon > 0$ there exist constants $L_1 > L_2 > 0$ which depend only on $|x|$, $|y|$, and ϵ such that for each solution $v : [T_1, T_2] \to R^n$ of the problem (P) and each $\tau \in [T_1 + L_1, T_2 - L_1]$, the set $\{v(t) : \ t \in [\tau, \tau + L_2]\}$ is equal to a set $H(f)$ up to ϵ in the Hausdorff metric. Here $H(f) \subset R^n$ is a compact set depending only on the integrand f. In Sects. 3.6–3.10, for the same class of integrands, we prove a strong version of the turnpike property stated above which was obtained in [50]. In this strong version instead of the inequality

$$\mathrm{dist}(\{v(t) : \ t \in [\tau, \tau + L_2]\}, H(f)) \le \epsilon$$

(here $\mathrm{dist}(\cdot, \cdot)$ is the distance in the Hausdorff metric), we obtain the inequality

$$|v(\tau + t) - w(s + t)| \le \epsilon \text{ for all } t \in [0, L_2],$$

where $w : R^1 \to R^n$ is a function depending only on f and the constant s depends on τ. In other words the restriction of the function v to the interval $[\tau, \tau + L_2]$ is equal to a translation of the function w up to ϵ, and this property holds for each $\tau \in [L_1, T - L_1]$. This result (Theorem 3.14) is stated in Sect. 3.6 and is proved in Sects. 3.7–3.9.

In Chap. 4 we study the structure of approximate solutions of variational problems with integrands $f : R^n \times R^n \to R^1$ which are convex functions on $R^n \times R^n$. The main goal of this chapter is to study the structure of approximate solutions of the problems (P) in the regions $[T_1, T_1 + L]$ and $[T_2 - L, T_2]$ (see the definition of the turnpike property). The results of this chapter, which were obtained in [52], provide the full description of the structure of solutions of the problems (P).

2

Nonautonomous Problems

In this chapter we consider the question how to verify if an integrand possesses the turnpike property and a trajectory X is its turnpike. We introduce two properties (P1) and (P2) and show that the integrand has the turnpike property if and only if it possesses properties (P1) and (P2). The property (P2) means that all approximate solutions of the corresponding infinite horizon variational problem have the same asymptotic behavior while the property (P1) means that any approximate solution on a sufficiently large interval is close to the turnpike at some point.

2.1 Preliminaries and Main Results

In this chapter, we study the structure of solutions of the variational problems:

$$\int_{T_1}^{T_2} f(t, z(t), z'(t))dt \to \min,\ z(T_1) = x,\ z(T_2) = y; \tag{P}$$

$$z : [T_1, T_2] \to R^n \text{ is an absolutely continuous (a.c.) function,}$$

where $T_1 \geq 0$, $T_2 > T_1$, $x, y \in R^n$, and $f : [0, \infty) \times R^n \times R^n \to R^1$ belong to a space of integrands described below.

Denote by $|\cdot|$ the Euclidean norm in R^n. Suppose that a real number a is positive and that $\psi : [0, \infty) \to [0, \infty)$ is an increasing function for which $\psi(t) \to +\infty$ as $t \to \infty$. We denote by $\mathcal{M}$ the set of all continuous functions $f : [0, \infty) \times R^n \times R^n \to R^1$ such that the following assumptions hold:

A(i) The function f is bounded on $[0, \infty) \times E$ for any bounded set $E \subset R^n \times R^n$.

A(ii) $f(t, x, u) \geq \max\{\psi(|x|), \psi(|u|)|u|\} - a$ for all $(t, x, u) \in [0, \infty) \times R^n \times R^n$.

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A(iii) For each pair of positive numbers M, ϵ, there exists a pair of positive numbers Γ, δ such that for each $t \in [0, \infty)$ and each $u, x_1, x_2 \in R^n$ satisfying

$$|x_i| \le M, \ i = 1, 2, \ |u| \ge \Gamma, \quad |x_1 - x_2| \le \delta,$$

we have

$$|f(t, x_1, u) - f(t, x_2, u)| \le \epsilon \max\{f(t, x_1, u), f(t, x_2, u)\}.$$

A(iv) For each pair of positive numbers M, ϵ, there exists a positive number δ such that for each $t \in [0, \infty)$ and each $u_1, u_2, x_1, x_2 \in R^n$ satisfying

$$|x_i|, |u_i| \le M, \ i = 1, 2, \quad \max\{|x_1 - x_2|, |u_1 - u_2|\} \le \delta,$$

we have

$$|f(t, x_1, u_1) - f(t, x_2, u_2)| \le \epsilon.$$

The space of integrands $\mathcal{M}$ was introduced in [46]. In [42, 44] and in Chaps. 1 and 2 of [51], we studied the subset of the set $\mathcal{M}$ which consists of all $f \in \mathcal{M}$ satisfying the following assumptions:
the function $f(t, x, \cdot) : R^n \to R^1$ is convex for any $(t, x) \in [0, \infty) \times R^n$;

for each pair of positive numbers M, ϵ there exists a pair of positive numbers Γ, δ such that for each $t \in [0, \infty)$ and each $u_1, u_2, x_1, x_2 \in R^n$ satisfying

$$|x_i| \le M, \ |u_i| \ge \Gamma, \ i = 1, 2, \quad \max\{|x_1 - x_2|, |u_1 - u_2|\} \le \delta,$$

we have

$$|f(t, x_1, u_1) - f(t, x_2, u_2)| \le \epsilon \max\{f(t, x_1, u_1), f(t, x_2, u_2)\}$$

(see A(iii)).

We denote this subset by $\mathcal{M}_{co}$.

It is easy to show that an integrand $f = f(t, x, u) \in C^1([0, \infty) \times R^n \times R^n)$ belongs to $\mathcal{M}$ if f satisfies assumption A(ii) and if $\sup\{|f(t, 0, 0)| : \ t \in [0, \infty)\} < \infty$ and also there exists an increasing function $\psi_0 : \ [0, \infty) \to [0, \infty)$ such that

$$\sup\{|\partial f/\partial x(t, x, u)|, \ |\partial f/\partial u(t, x, u)|\} \le \psi_0(|x|)(1 + \psi(|u|)|u|)$$

for each $t \in [0, \infty)$ and each $x, u \in R^n$.

We equip the set $\mathcal{M}$ with the uniformity determined by the following base:

$$E(N, \epsilon, \lambda) = \{(f, g) \in \mathcal{M} \times \mathcal{M} : |f(t, x, u) - g(t, x, u)| \le \epsilon \tag{2.1}$$

$$\text{for each } t \in [0, \infty) \text{ and each } x, u \in R^n \text{ satisfying } |x|, |u| \le N$$

$$\text{and } (|f(t, x, u)| + 1)(|g(t, x, u)| + 1)^{-1} \in [\lambda^{-1}, \lambda]$$

$$\text{for each } t \in [0, \infty) \text{ and each } x, u \in R^n \text{ satisfying } |x| \le N\},$$

where $N > 0$, $\epsilon > 0$, and $\lambda > 1$.

It is easy to see that the space $\mathcal{M}$ equipped with this uniformity is metrizable (by a metric ρ_w). We show in [46] that the metric space $(\mathcal{M}, \rho_w)$ is complete. We consider the topological space $\mathcal{M}$ with a topology induced by the metric ρ_w.

In this chapter, we consider integral functionals

$$I^f(T_1, T_2, x) = \int_{T_1}^{T_2} f(t, x(t), x'(t))dt \tag{2.2}$$

where $f \in \mathcal{M}$, $0 \le T_1 < T_2 < \infty$, and $x : [T_1, T_2] \to R^n$ is an a.c. function.

For any $f \in \mathcal{M}$, any pair of points $y, z \in R^n$ and any pair of real numbers T_1, T_2 satisfying $0 \le T_1 < T_2$ put

$$U^f(T_1, T_2, y, z) = \inf\{I^f(T_1, T_2, x) : \ x : \ [T_1, T_2] \to R^n \tag{2.3}$$
$$\text{is an a.c. function satisfying } x(T_1) = y, \ x(T_2) = z\}.$$

Evidently, the value $U^f(T_1, T_2, y, z)$ is finite for each function $f \in \mathcal{M}$, each pair of points $y, z \in R^n$ and each pair of real numbers T_1, T_2 satisfying $0 \le T_1 < T_2$.

Let $f \in \mathcal{M}$. We say that a locally absolutely continuous (a.c.) function $x : [0, \infty) \to R^n$ is an (f)-*good function* [51] if for any a.c function $y : [0, \infty) \to R^n$, there exists a real number M_y depending on y which satisfies

$$I^f(0, T, y) \ge M_y + I^f(0, T, x) \text{ for all positive numbers } T. \tag{2.4}$$

In Sect. 2.3, we will prove the following result which was obtained in [47].

Proposition 2.1. *Assume that $f \in \mathcal{M}$ and that $x : [0, \infty) \to R^n$ is a bounded a.c. function. Then the function x is (f)-good if and only if there exists a positive number M which satisfies*

$$I^f(0, T, x) \le U^f(0, T, x(0), x(T)) + M \textit{ for all } T > 0.$$

Let $f \in \mathcal{M}$. We say that f possesses the turnpike property, or briefly TP, if there exists a bounded continuous function $X_f : [0, \infty) \to R^n$ which satisfies the following condition:

For each pair of positive numbers K, ϵ, there exists a pair of positive constants δ, L such that for each pair of points $x, y \in R^n$ satisfying $|x|, |y| \le K$; each pair of real numbers $T_1 \ge 0$, $T_2 \ge T_1 + 2L$; and each a.c. function $v : [T_1, T_2] \to R^n$ satisfying

$$v(T_1) = x, \ v(T_2) = y, \ I^f(T_1, T_2, v) \le U^f(T_1, T_2, x, y) + \delta,$$

we have

$$|v(t) - X_f(t)| \le \epsilon \text{ for all } t \in [T_1 + L, T_2 - L].$$

The function X_f is called the turnpike of f.

In [45], we establish the existence of an everywhere dense subset $\mathcal{F} \subset \mathcal{M}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}$ such that any integrand $f \in \mathcal{F}$ possesses TP. This result is a generalization of the main result of [44] which was obtained for the space $\mathcal{M}_{co}$. Note that there are integrands which belong to the space $\mathcal{M}$ and do not possess TP [44]. Therefore, the main result of [45] cannot be essentially improved. Nevertheless, some questions are still open. Namely, let $f \in \mathcal{M}$ and let $X : [0, \infty) \to R^n$ be a bounded continuous function. We are interested to answer the question how to verify if the integrand f has TP and X is its turnpike.

In [47], we introduce two properties (P1) and (P2) and show that f possesses the turnpike property if and only if f has the properties (P1) and (P2). According to the property (P2), all (f)-good functions have the same asymptotic behavior. The property (P1) means that if an a.c. function $v : [0, T] \to R^n$ is an approximate solution and T is large enough, then $v(\tau)$ is close to $X(\tau)$ with some $\tau \in [0, T]$.

In Sect. 2.6, we will prove the following theorem which was established in [47].

Theorem 2.2. *Assume that $f \in \mathcal{M}$ and that $X_f : [0, \infty) \to R^n$ is a bounded continuous function. Then f possesses the turnpike property with X_f being the turnpike if and only if the following two properties hold:*

(P1) For each pair of positive numbers K, ϵ, there exists a a pair of positive numbers γ, l such that for each nonnegative number T and each a.c. function $w : [T, T + l] \to R^n$ which satisfies

$$|w(T)|, |w(T + l)| \le K,\ I^f(T, T + l, w) \le U^f(T, T + l, w(T), w(T + l)) + \gamma,$$

there exists a number $\tau \in [T, T + l]$ such that $|X_f(\tau) - w(\tau)| \le \epsilon$.

(P2) For each (f)-good function $v : [0, \infty) \to R^n$,

$$|v(t) - X_f(t)| \to 0 \text{ as } t \to \infty.$$

The following strong turnpike property was introduced in [49].

Let $f \in \mathcal{M}$. We say that the integrand f possesses the *strong turnpike property*, or briefly STP, if there exists a bounded a.c. function $X_f : [0, \infty) \to R^n$ such that the following property holds:

For each pair of positive numbers K, ϵ, there exists a pair of positive constants δ, L such that for each $T_1 \ge 0$, $T_2 \ge T_1 + 2L$ and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies

$$|v(T_1)|, |v(T_2)| \le K,\ I^f(T_1, T_2, v) \le U^f(T_1, T_2, v(T_1), v(T_2)) + \delta$$

we have:

(i) There exist $\tau_1 \in [T_1, T_1 + L]$ and $\tau_2 \in [T_2 - L, T_2]$ such that

$$|v(t) - X_f(t)| \le \epsilon$$

for all $t \in [\tau_1, \tau_2]$.

(ii) If $|v(T_1) - X_f(T_1)| \le \delta$, then $\tau_1 = T_1$, and if $|v(T_2) - X_f(T_2)| \le \delta$, then $\tau_2 = T_2$.

The function X_f is called the turnpike of f.

It is clear that if in the definition of STP condition (ii) is not assumed, then the integrand f possesses the turnpike property defined above.

In [49], we proved the following proposition which is useful in our study of STP.

Proposition 2.3. *Assume that $f \in \mathcal{M}$ and that the function $f(t,x,\cdot) : R^n \to R^1$ is convex for each $(t,x) \in [0,\infty) \times R^n$. Then for any point $z \in R^n$, there exists a bounded (f)-good function $Z : [0,\infty) \to R^n$ which satisfies $Z(0) = z$ and*

$$I^f(0,T,Z) = U^f(0,T,Z(0),Z(T)) \text{ for all } T > 0.$$

In this chapter, we use the following overtaking optimality criterion which was introduced in the economic literature [37] and used in the infinite horizon optimal control theory [12, 51].

Assume that $f \in \mathcal{M}$. An a.c. function $x : [0,\infty) \to R^n$ is called (f)-overtaking optimal if for each a.c. function $y : [0,\infty) \to R^n$ which satisfies $y(0) = x(0)$, we have

$$\limsup_{T\to\infty}[I^f(0,T,x) - I^f(0,T,y)] \le 0.$$

Let $f \in \mathcal{M}$ and let $X : [0,\infty) \to R^n$ be a bounded a.c. function. We are interested to answer the question how to verify if the integrand f possesses the STP and X is its turnpike. These three properties, (Q1), (Q2), and (Q3), were introduced in [49] where we show that the integrand f possesses STP if and only if the properties (Q1), (Q2), and (Q3) hold with the function f. Property (Q1) shows that all (f)-good functions have the same asymptotic behavior. According to property (Q2), X is a unique (f)-overtaking optimal function whose value at zero is $X(0)$. Property (Q3) means that if an a.c. function $v : [0,T] \to R^n$ is an approximate solution and T is large enough, then $v(\tau)$ belongs to a small neighborhood of $X(\tau)$ with some $\tau \in [0,T]$.

In this chapter, we prove the following theorem which was established in [49].

Theorem 2.4. *Suppose that $f \in \mathcal{M}$, the function $f(t,x,\cdot) : R^n \to R^1$ is convex for each $(t,x) \in [0,\infty) \times R^n$, and $X_f : [0,\infty) \to R^n$ is a bounded a.c. function. Then the integrand f possesses STP with the function X_f being the turnpike if and only if the following three properties hold:*

(Q1) For each pair of (f)-good functions $v_1, v_2 : [0,\infty) \to R^n$,

$$|v_1(t) - v_2(t)| \to 0 \text{ as } t \to \infty.$$

(Q2) X_f *is an* (f)*-overtaking optimal function, and if an* (f)*-overtaking optimal function* $v : [0, \infty) \to R^n$ *satisfies* $v(0) = X_f(0)$*, then* $v = X_f$*.*

(Q3) For each pair of positive numbers K, ϵ*, there exists a pair of positive numbers* γ, l *such that for each nonnegative number* T *and each a.c. function* $w : [T, T + l] \to R^n$ *for which*

$$|w(T)|, |w(T + l)| \le K$$

and

$$I^f(T, T + l, w) \le U^f(T, T + l, w(T), w(T + l)) + \gamma,$$

there exists a real number $\tau \in [T, T + l]$ *satisfying*

$$|X_f(\tau) - v(\tau)| \le \epsilon.$$

2.2 Auxiliary Results

In order to study the turnpike properties, we need the following result established in [46].

Proposition 2.5. *For each* $h \in \mathcal{M}$*, each* $\delta \in (0, 1)$*, and each* $z \in R^n$*, there exists an* (h)*-good function* $Z^h_\delta : [0, \infty) \to R^n$ *satisfying* $Z^h_\delta(0) = z$ *such that the following assertions hold:*

1. *Let* $f \in \mathcal{M}$*,* $\epsilon \in (0, 1)$*, and* $z \in R^n$*, and let* $y : [0, \infty) \to R^n$ *be an a.c. function. Then one of the following properties holds:*
   (i) $I^f(0, T, y) - I^f(0, T, Z^f_\epsilon) \to \infty$ *as* $T \to \infty$*.*
   (ii) $\sup\{|I^f(0, T, y) - I^f(0, T, Z^f_\epsilon)| : T \in (0, \infty)\} < \infty$ *and*
   $$\sup\{|y(t)| : t \in [0, \infty)\} < \infty.$$
2. *For each* $f \in \mathcal{M}$*, and each positive number* M*, there exist a neighborhood* U *of* f *in* $\mathcal{M}$ *and a number* $Q > 0$ *such that*
   $$\sup\{|Z^g_\epsilon(t)| : t \in [0, \infty)\} \le Q$$
   for each $g \in U$*, each* $\epsilon \in (0, 1)$ *and each* $z \in R^n$ *satisfying* $|z| \le M$*.*
3. *For each* $f \in \mathcal{M}$ *and each positive number* M*, there exist a neighborhood* U *of* f *in* $\mathcal{M}$ *and a number* $Q > 0$ *such that for each* $g \in U$*, each* $z \in R^n$ *satisfying* $|z| \le M$*, each* $\epsilon \in (0, 1)$*, each* $T_1 \ge 0$*,* $T_2 > T_1$*, and each a.c. function* $y : [T_1, T_2] \to R^n$ *satisfying* $|y(T_1)| \le M$*, the following relation holds:*
   $$I^g(T_1, T_2, Z^g_\epsilon) \le I^g(T_1, T_2, y) + Q.$$

4. For each $f \in \mathcal{M}$, each $\epsilon > 0$, each $z \in R^n$, each $T_1 \geq 0$, and each $T_2 > T_1$,

$$I^f(T_1, T_2, Z_\epsilon^f) \leq U^f(T_1, T_2, Z_\epsilon^f(T_1), Z_\epsilon^f(T_2)) + \epsilon.$$

5. For each $f \in \mathcal{M}$, each $z \in R^n$, and each integer $i \geq 0$,

$$Z_{\epsilon_1}^f(i) = Z_{\epsilon_2}^f(i) \text{ for each } \epsilon_1, \epsilon_2 \in (0, 1).$$

Proposition 2.5 is a generalization of an analogous result of [42] established for the space $\mathcal{M}_{co}$. There we show that for each $f \in \mathcal{M}_{co}$ and each $z \in R^n$,

$$Z_{\epsilon_1}^f = Z_{\epsilon_2}^f \text{ for each } \epsilon_1, \epsilon_2 \in (0, 1)$$

and that

$$U^f(T_1, T_2, Z_\epsilon^f(T_1), Z_\epsilon^f(T_2)) = I^f(T_1, T_2, Z_\epsilon^f)$$

for each $T_1 \geq 0$, $T_2 > T_1$ and each $\epsilon \in (0, 1)$.

The following useful results were established in [46].

Proposition 2.6. *For each $f \in \mathcal{M}$, there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ and a number $M > 0$ such that for each $g \in \mathcal{U}$ and each (g)-good function $x : [0, \infty) \to R^n$, the relation $\limsup_{t\to\infty} |x(t)| < M$ holds.*

Proposition 2.7. *Let $f \in \mathcal{M}$ and M_1, M_2, c be positive numbers. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ and a number $S > 0$ such that for each $g \in \mathcal{U}$, each $T_1 \in [0, \infty)$, and each $T_2 \in [T_1 + c, \infty)$, the following property holds:*

For each $x, y \in R^n$ satisfying $|x|, |y| \leq M_1$ and each a.c. function $v : [T_1, T_2] \to R^n$ satisfying

$$v(T_1) = x,\ v(T_2) = y,\ I^g(T_1, T_2, v) \leq U^g(T_1, T_2, x, y) + M_2,$$

the following relation holds:

$$|v(t)| \leq S,\ t \in [T_1, T_2].$$

Proposition 2.8. *Let $f \in \mathcal{M}$, $0 < c_1 < c_2 < \infty$, $c_3, \epsilon > 0$. Then there exists a neighborhood V of f in $\mathcal{M}$ such that for each $g \in V$, each $T_1, T_2 \geq 0$ satisfying $T_2 - T_1 \in [c_1, c_2]$, and each $y, z \in R^n$ satisfying $|y|, |z| \leq c_3$, the relation $|U^f(T_1, T_2, y, z) - U^g(T_1, T_2, y, z)| \leq \epsilon$ holds.*

Proposition 2.9. *Let $f \in \mathcal{M}$, $0 < c_1 < c_2 < \infty$, $c_3 > 0$. Then there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{M}$ such that the set*

$$\{U^g(T_1, T_2, z_1, z_2) :\ g \in \mathcal{U},\ T_1 \in [0, \infty),\ T_2 \in [T_1 + c_1, T_1 + c_2],$$
$$z_1, z_2 \in R^n,\ |z_i| \leq c_3,\ i = 1, 2\}$$

is bounded.

Proposition 2.10. *Let $f \in \mathcal{M}$, $0 < c_1 < c_2 < \infty$, $D, \epsilon > 0$. Then there exists a neighborhood V of f in $\mathcal{M}$ such that for each $g \in V$, each $T_1, T_2 \geq 0$ satisfying $T_2 - T_1 \in [c_1, c_2]$, and each a.c. function $x : [T_1, T_2] \to R^n$ satisfying*

$$\min\{I^f(T_1, T_2, x),\ I^g(T_1, T_2, x)\} \leq D,$$

the relation $|I^f(T_1, T_2, x) - I^g(T_1, T_2, x)| \leq \epsilon$ holds.

Proposition 2.11. *Let $f \in \mathcal{M}$, $0 < c_1 < c_2 < \infty$, $M, \epsilon > 0$. Then there exists $\delta > 0$ such that for each $T_1, T_2 \geq 0$ satisfying $T_2 - T_1 \in [c_1, c_2]$ and each $y_1, y_2, z_1, z_2 \in R^n$ satisfying*

$$|y_i|, |z_i| \leq M,\ i = 1, 2, \quad |y_1 - y_2|, |z_1 - z_2| \leq \delta,$$

the relation $|U^f(T_1, T_2, y_1, z_1) - U^f(T_1, T_2, y_2, z_2)| \leq \epsilon$ holds.

Proposition 2.12. *Let $M_1, \epsilon > 0$, $0 < \tau_0 < \tau_1$. Then there exists $\delta > 0$ such that for each $f \in \mathcal{M}$, each $T_1 \in [0, \infty)$, $T_2 \in [T_1 + \tau_0, T_1 + \tau_1]$, each a.c. function $x : [T_1, T_2] \to R^n$ satisfying $I^f(T_1, T_2, x) \leq M_1$, and each $t_1, t_2 \in [T_1, T_2]$ which satisfy $|t_2 - t_1| \leq \delta$, the relation $|x(t_1) - x(t_2)| \leq \epsilon$ holds.*

2.3 Proof of Proposition 2.1

In view of Proposition 2.5 there is a bounded (f)-good function $z : [0, \infty) \to R^n$ which satisfies

$$z(0) = x(0),$$
$$I^f(0, T, z) \leq U^f(0, T, z(0), z(T)) + 1/2 \tag{2.5}$$

for each nonnegative number T and such that the following properties hold:

(i) For any a.c. function $y : [0, \infty) \to R^n$ either

$$\sup\{|I^f(0, T, y) - I^f(0, T, z)| :\ T \in (0, \infty)\} < \infty$$

or

$$I^f(0, T, y) - I^f(0, T, z) \to \infty \text{ as } T \to \infty.$$

(ii) There exists a positive number Q such that for each positive number T and each a.c. function $y : [0, T] \to R^n$ which satisfies $y(0) = x(0)$, we have

$$I^f(0, T, y) + Q \geq I^f(0, T, z).$$

By the property (i) and the definition of an (f)-good function, we have that the following property holds:

(iii) The function x is (f)-good if and only if

$$\sup\{|I^f(0,T,x) - I^f(0,T,z)| : T \in (0,\infty)\} < \infty. \tag{2.6}$$

Assume that the function x is (f)-good. Then by (iii), inequality (2.6) is true. Fix a real number

$$\Delta > \sup\{|I^f(0,T,x) - I^f(0,T,z)| : T \in (0,\infty)\}. \tag{2.7}$$

It follows from inequality (2.7) and the property (ii) that for each positive number T and each a.c. function $y : [0,T] \to R^n$ which satisfies $y(0) = x(0)$, we have

$$I^f(0,T,y) \geq I^f(0,T,z) - Q \geq I^f(0,T,x) - \Delta - Q.$$

This implies that for any positive number T the following inequality is true:

$$U^f(0,T,x(0),x(T)) \geq I^f(0,T,x) - \Delta - Q.$$

Now assume that there exists a positive number M for which

$$I^f(0,T,x) \leq U^f(0,T,x(0),x(T)) + M \text{ for all } T > 0. \tag{2.8}$$

We claim that x is an (f)-good function.

Fix a real number

$$\Delta > \sup\{|z(t)| : t \in [0,\infty)\} + \sup\{|x(t)| : t \in [0,\infty)\}. \tag{2.9}$$

Let $T > 2$. Consider an a.c. function $y : [0,T] \to R^n$ such that

$$y(t) = x(0) + t(z(1) - x(0)),\ t \in [0,1],\ y(t) = z(t),\ t \in [1, T-1],$$

$$y(t) = z(T-1) + (t-(T-1))(x(T) - z(T-1)),\ t \in [T-1,T]. \tag{2.10}$$

It follows from (2.8)–(2.10) and the assumptions A(i) and A(ii) that

$$\begin{aligned}
I^f(0,T,x) &\leq I^f(0,T,y) + M = M + I^f(0,1,y) + I^f(1,T-1,y) + I^f(T-1,T,y) \\
&\leq M + I^f(0,1,y) + I^f(T-1,T,y) + I^f(0,T,z) - I^f(0,1,z) \\
&\quad - I^f(T-1,T,z) \leq M + I^f(0,T,z) + 2a + I^f(0,1,y) + I^f(T-1,T,y) \\
&\leq M + 2a + I^f(0,T,z) \\
&\quad + 2\sup\{f(t,u,v) : t \in [0,\infty),\ u,v \in R^n,\ |u|,|v| \leq 2\Delta\}.
\end{aligned}$$

Since the inequality above is true for all $T > 2$, the properties (i) and (iii) imply that the function x is (f)-good. This completes the proof of the proposition.

2.4 TP Implies (P1) and (P2)

First, we prove the following auxiliary result.

Lemma 2.13. *Assume that $f \in \mathcal{M}$, $v : [0,\infty) \to R^n$ is an (f)-good function and that ϵ is a positive number. Then there exists a positive number T_ϵ such that for each pair of numbers $T_1 \ge T_\epsilon$, $T_2 > T_1$, the following inequality holds:*

$$I^f(T_1, T_2, v) \le U^f(T_1, T_2, v(T_1), v(T_2)) + \epsilon.$$

Proof. Let us assume the contrary. Then the following property holds:

For each positive number τ there exists a pair of numbers $\tau_1 \ge \tau$, $\tau_2 > \tau_1$ which satisfies

$$I^f(\tau_1, \tau_2, v) > U^f(\tau_1, \tau_2, v(\tau_1), v(\tau_2)) + \epsilon.$$

It follows from the property above that there exists a pair of sequences $\{t_i\}_{i=0}^\infty$, $\{s_i\}_{i=0}^\infty \subset (0,\infty)$ such that for all natural numbers i we have

$$t_i < s_i < t_{i+1} - 4,\ I^f(t_i, s_i, v) > U^f(t_i, s_i, v(t_i), v(s_i)) + \epsilon. \tag{2.11}$$

For each natural number i, there exists an a.c. function $u_i : [t_i, s_i] \to R^n$ which satisfies

$$u_i(t_i) = v(t_i),\ u_i(s_i) = v(s_i),\ I^f(t_i, s_i, u_i) \le U^f(t_i, s_i, v(t_i), v(s_i)) + \epsilon/2. \tag{2.12}$$

Define an a.c. function $u : [0,\infty) \to R^n$ as follows:

$$\begin{aligned} &u(t) = u_i(t),\ t \in [t_i, s_i],\ i = 1, 2, \dots, \\ &u(t) = v(t),\ t \in [0,\infty) \setminus \cup_{i=1}^\infty [t_i, s_i]. \end{aligned} \tag{2.13}$$

It is easy to see that the function u is well defined. Since the function v is (f)-good there exists a positive number M_0 which satisfies

$$I^f(0, T, u) + M_0 \ge I^f(0, T, v) \text{ for all } T \in (0,\infty). \tag{2.14}$$

On the other hand (2.11)–(2.13) imply that for any integer $k \ge 1$, we have

$$I^f(0, s_k, u) - I^f(0, s_k, v) = \sum_{i=1}^k [I^f(t_i, s_i, u_i) - I^f(t_i, s_i, v)]$$

$$\le \sum_{i=1}^k [U^f(t_i, s_i, v(t_i), v(s_i)) + \epsilon/2 - U^f(t_i, s_i, v(t_i), v(s_i)) - \epsilon]$$

$$= -k\epsilon/2 \to -\infty \text{ as } k \to \infty,$$

a contradiction. The contradiction we have reached completes the proof of Lemma 2.13.

Assume that $f \in \mathcal{M}$, $X_f : [0, \infty) \to R^n$ is a bounded continuous function, f possesses the turnpike property, and X_f is the turnpike of f. Evidently, the property (P1) holds. We claim that property (P2) also holds.

Let $v : [0, \infty) \to R^n$ be an (f)-good function, and let ϵ be a positive number. Proposition 2.6 implies that there exists a positive number c_0 for which

$$|v(t)| \le c_0 \text{ for all } t \in [0, \infty). \tag{2.15}$$

Since f possesses the turnpike property and X_f is the turnpike of f there exists a pair of positive numbers δ_0, L_0 for which the following property holds:

(i) For each pair of numbers $T_1 \ge 0$, $T_2 \ge T_1 + 2L_0$ and each a.c. function $u : [T_1, T_2] \to R^n$ such that

$$|u(T_1)|, |u(T_2)| \le c_0 + 1, \; I^f(T_1, T_2, u) \le U^f(T_1, T_2, u(T_1), u(T_2)) + \delta_0,$$

we have $|u(t) - X_f(t)| \le \epsilon$ for all $t \in [T_1 + L_0, T_2 - L_0]$.

Lemma 2.13 implies that there exists a real number $S_0 > 4$ such that for each pair of numbers $T_1 \ge S_0$, $T_2 > T_1$, we have

$$I^f(T_1, T_2, v) \le U^f(T_1, T_2, v(T_1), v(T_2)) + \delta_0. \tag{2.16}$$

Let $\tau > S_0 + L_0$. Put

$$T_1 = \tau - L_0, \; T_2 = \tau + 2L_0. \tag{2.17}$$

In view of (2.17) and the definition of S_0, inequality (2.16) holds. By (2.15)–(2.17) and the property (i), we have

$$|v(t) - X_f(t)| \le \epsilon, \; t \in [T_1 + L_0, T_2 - L_0] = [\tau, \tau + L_0].$$

Hence $|v(t) - X_f(t)| \le \epsilon$ for all numbers $\tau > S_0 + L_0$. This implies that $|v(t) - X_f(t)| \to 0$ as $t \to \infty$ and that the property (P2) holds as claimed.

2.5 The Basic Lemma for Theorem 2.2

For each $x \in R^n$ and each nonempty subset $A \subset R^n$, put

$$d(x, A) = \inf\{|x - y| : \; y \in A\}.$$

Lemma 2.14. *Assume that $f \in \mathcal{M}$, $X_f : [0, \infty) \to R^n$ is an (f)-good function and that*

$$\lim_{t \to \infty} |v(t) - X_f(t)| = 0 \tag{2.18}$$

for each (f)-good function $v : [0, \infty) \to R^n$. Then for each positive number ϵ, there exists a pair of positive numbers T_0, δ_0 such that the following property holds:

(P3) For each pair of numbers $T_1 \ge T_0$, $T_2 \ge T_1 + 1$ and each a.c. function $u : [T_1, T_2] \to R^n$ which satisfies

$$|u(T_i) - X_f(T_i)| \le \delta_0, \ i = 1, 2 \tag{2.19}$$

and

$$I^f(T_1, T_2, u) \le U^f(T_1, T_2, u(T_1), u(T_2)) + \delta_0, \tag{2.20}$$

the following inequality is valid:

$$|u(t) - X_f(t)| \le \epsilon \text{ for all } t \in [T_1, T_2]. \tag{2.21}$$

Proof. Since the function X_f is (f)-good Proposition 2.6 implies that X_f is bounded. Fix a real number

$$M_0 > \sup\{|X_f(t)| : \ t \in [0, \infty)\} + 1. \tag{2.22}$$

Let $\epsilon > 0$ be given. We claim that there exists a pair of positive numbers T_0, δ_0 for which the property (P3) holds.

Let us assume the contrary. Then the following property holds:

(P4) For each pair of positive numbers S, δ there exist $T_1 \ge S$, $T_2 \ge T_1 + 1$, and an a.c. function $u : [T_1, T_2] \to R^n$ which satisfy

$$|u(T_i) - X_f(T_i)| \le \delta, \ i = 1, 2,$$

$$I^f(T_1, T_2, v) \le U^f(T_1, T_2, v(T_1), v(T_2)) + \delta$$

and

$$\sup\{|u(t) - X_f(t)| : \ t \in [T_1, T_2]\} > \epsilon.$$

In view of Proposition 2.11 for each natural number i there is

$$\delta_i \in (0, 4^{-i}) \tag{2.23}$$

for which the following property holds:

(a) For each pair of numbers $T_1 \ge 0$, $T_2 \in [T_1 + 8^{-1}, T_1 + 8]$ and each $x_1, x_2, y_1, y_2 \in R^n$ such that

$$|x_j|, |y_j| \le M_0 + 2, \ j = 1, 2, \ |x_j - y_j| \le 2\delta_i, \ i = 1, 2,$$

we have

$$|U^f(T_1, T_2, x_1, x_2) - U^f(T_1, T_2, y_1, y_2)| \le 4^{-i}.$$

By Lemma 2.13 and the property (P4), we define by induction sequences $\{T_{i1}\}_{i=1}^{\infty}$, $\{T_{i2}\}_{i=1}^{\infty} \subset (0, \infty)$ and a sequence of a.c. functions

$$v_i : [T_{i1}, T_{i2}] \to R^n$$

such that $T_{11} \geq 4$ and that for each natural number integer i, we have

$$T_{i1} + 1 \leq T_{i2} < T_{(i+1)1} - 8, \tag{2.24}$$

$$I^f(T_{i1}, T_{i2}, X_f) \leq U^f(T_{i1}, T_{i2}, X_f(T_{i1}), X_f(T_{i2})) + 4^{-i}, \tag{2.25}$$

$$|v_i(T_{i1}) - X_f(T_{i1})| \leq \delta_i, \ |v_i(T_{i2}) - X_f(T_{i2})| \leq \delta_i, \tag{2.26}$$

$$I^f(T_{i1}, T_{i2}, v_i) \leq U^f(T_{i1}, T_{i2}, v_i(T_{i1}), v_i(T_{i2})) + \delta_i \tag{2.27}$$

and

$$\sup\{|v_i(t) - X_f(t)| : \ t \in [T_{i1}, T_{i2}]\} > \epsilon. \tag{2.28}$$

Let a natural number i be given. There exists an a.c. function $\tilde{v}_i : [T_{i1} - 1, T_{i2} + 1] \to R^n$ which satisfies

$$\tilde{v}_i(t) = v_i(t), \ t \in [T_{i1}, T_{i2}], \tag{2.29}$$

$$\tilde{v}_i(T_{i1} - 1) = X_f(T_{i1} - 1), \ \tilde{v}_i(T_{i2} + 1) = X_f(T_{i2} + 1), \tag{2.30}$$

$$I^f(T_{i1} - 1, T_{i1}, \tilde{v}_i) \leq U^f(T_{i1} - 1, T_{i1}, X_f(T_{i1} - 1), v_i(T_{i1})) + \delta_i, \tag{2.31}$$

$$I^f(T_{i2}, T_{i2} + 1, \tilde{v}_i) \leq U^f(T_{i2}, T_{i2} + 1, v_i(T_{i2}), X_f(T_{i2} + 1)) + \delta_i. \tag{2.32}$$

Let a natural number i be given. We estimate

$$I^f(T_{i1} - 1, T_{i2} + 1, \tilde{v}_i) - I^f(T_{i1} - 1, T_{i2} + 1, X_f).$$

It follows from (2.32), (2.31), (2.27), and (2.29) that

$$\begin{aligned}
&I^f(T_{i1} - 1, T_{i2} + 1, \tilde{v}_i) - I^f(T_{i1} - 1, T_{i2} + 1, X_f)\\
&= I^f(T_{i1} - 1, T_{i1}, \tilde{v}_i) - I^f(T_{i1} - 1, T_{i1}, X_f) + I^f(T_{i1}, T_{i2}, v_i)\\
&-I^f(T_{i1}, T_{i2}, X_f) + I^f(T_{i2}, T_{i2} + 1, \tilde{v}_i) - I^f(T_{i2}, T_{i2} + 1, X_f)\\
&\leq U^f(T_{i1} - 1, T_{i1}, X_f(T_{i1} - 1), v_i(T_{i1})) + \delta_i\\
&-U^f(T_{i1} - 1, T_{i1}, X_f(T_{i1} - 1), X_f(T_{i1}))\\
&+U^f(T_{i1}, T_{i2}, v_i(T_{i1}), v_i(T_{i2})) + \delta_i\\
&-U^f(T_{i1}, T_{i2}, X_f(T_{i1}), X_f(T_{i2}))\\
&+U^f(T_{i2}, T_{i2} + 1, v_i(T_{i2}), X_f(T_{i2} + 1)) + \delta_i\\
&- U^f(T_{i2}, T_{i2} + 1, X_f(T_{i2}), X_f(T_{i2} + 1)).
\end{aligned} \tag{2.33}$$

By property (a), the choice of δ_i and (2.23), (2.26), and (2.22),

$$|U^f(T_{i1}-1, T_{i1}, X_f(T_{i1}-1), v_i(T_{i1}))$$
$$-U^f(T_{i1}-1, T_{i1}, X_f(T_{i1}-1), X_f(T_{i1}))| \le 4^{-i},$$
$$|U^f(T_{i2}.T_{i2}+1, v_i(T_{i2}), X_f(T_{i_2}+1))$$
$$-U^f(T_{i2}, T_{i2}+1, X_f(T_{i2}), X_f(T_{i2}+1))| \le 4^{-i}.$$

Together with (2.33) these inequalities imply that

$$\begin{aligned} &I^f(T_{i1}-1, T_{i2}+1, \tilde{v}_i) - I^f(T_{i1}-1, T_{i2}+1, X_f) \\ &\le 3\delta_i + 2\cdot 4^{-i} + U^f(T_{i1}, T_{i2}, v_i(T_{i1}), v_i(T_{i2})) \\ &\quad - U^f(T_{i1}, T_{i2}, X_f(T_{i1}), X_f(T_{i2})). \end{aligned} \tag{2.34}$$

We estimate

$$U^f(T_{i1}, T_{i2}, v_i(T_{i1}), v_i(T_{i2})) - U^f(T_{i1}, T_{i2}, X_f(T_{i1}), X_f(T_{i2})).$$

There exists an a.c. function $w : [T_{i1}, T_{i2}] \to R^n$ which satisfies

$$\begin{aligned} &w(T_{i1}) = v_i(T_{i1}),\ w(T_{i2}) = v_i(T_{i2}), \\ &w(t) = X_f(t),\ t \in [T_{i1}+1/4,\ T_{i2}-1/4], \\ &I^f(T_{i1}, T_{i1}+1/4, w) \le U^f(T_{i1}, T_{i1}+1/4, v_i(T_{i1}), X_f(T_{i1}+1/4)) + \delta_i, \\ &I^f(T_{i2}-1/4, T_{i2}, w) \le \delta_i + U^f(T_{i2}-1/4, T_{i2}, X_f(T_{i2}-1/4), v_i(T_{i2})). \end{aligned} \tag{2.35}$$

In view of (2.26) and (2.22), the choice of δ_i and the property (a), we have

$$|U^f(T_{i1}, T_{i1}+1/4, v_i(T_{i1}), X_f(T_{i1}+1/4))$$
$$-U^f(T_{i1}, T_{i1}+1/4, X_f(T_{i1}), X_f(T_{i1}+1/4))| \le 4^{-i},$$
$$|U^f(T_{i2}-1/4, T_{i2}, X_f(T_{i2}-1/4), v_i(T_{i2}))$$
$$-U^f(T_{i2}-1/4, T_{i2}, X_f(T_{i2}-1/4), X_f(T_{i2}))| \le 4^{-i}.$$

Together with property (a); (2.35), (2.25), and (2.23) and the choice of δ_i these inequalities imply that

$$\begin{aligned} &U^f(T_{i1}, T_{i2}, v_i(T_{i1}), v_i(T_{i2})) - U^f(T_{i1}, T_{i2}, X_f(T_{i1}), X_f(T_{i2})) \\ &\le I^f(T_{i1}, T_{i2}, w) - I^f(T_{i1}, T_{i2}, X_f) + 4^{-i} \\ &= 4^{-i} + I^f(T_{i1}, T_{i2}+4^{-1}, w) + I^f(T_{i1}+4^{-1}, T_{i2}-4^{-1}, w) \\ &+I^f(T_{i2}-4^{-1}, T_{i2}, w) - [I^f(T_{i1}, T_{i1}+4^{-1}, X_f) + I^f(T_{i1}+4^{-1}, T_{i2}-4^{-1}, X_f) \\ &+I^f(T_{i2}-4^{-1}, T_{i2}, X_f)] = 4^{-i} + I^f(T_{i1}, T_{i1}+4^{-1}, w) + I^f(T_{i2}-4^{-1}, T_{i2}, w) \\ &-I^f(T_{i1}, T_{i1}+4^{-1}, X_f) - I^f(T_{i2}-4^{-1}, T_{i2}, X_f) \le 4^{-i} + \delta_i \\ &+U^f(T_{i1}, T_{i1}+4^{-1}, v_i(T_{i1}), X_f(T_{i1}+4^{-1})) \\ &+\delta_i + U^f(T_{i2}-4^{-1}, T_{i2}, X_f(T_{i2}-4^{-1}), v_i(T_{i2})) \\ &-U^f(T_{i1}, T_{i1}+4^{-1}, X_f(T_{i1}), X_f(T_{i1}+4^{-1})) \\ &-U^f(T_{i2}-4^{-1}, T_{i2}, X_f(T_{i2}-4^{-1}), X_f(T_{i2})) \end{aligned}$$

and

$$U^f(T_{i1}, T_{i2}, v_i(T_{i1}), v_i(T_{i2})) - U^f(T_{i1}, T_{i2}, X_f(T_{i1}), X_f(T_{i2})) < 5 \cdot 4^{-i}.$$

By this inequality and (2.34) and (2.23), we have

$$I^f(T_{i1} - 1, T_{i2} + 1, \tilde{v}_i) - I^f(T_{i1} - 1, T_{i2} + 1, X_f) \leq 5 \cdot 4^{-i} + 5 \cdot 4^{-i}. \quad (2.36)$$

Consider the function $u : [0, \infty) \to R^n$ defined as follows:

$$u(t) = \tilde{v}_i(t),\ t \in [T_{i1} - 1, T_{i2} + 1],\ i = 1, 2, \ldots,$$

$$u(t) = X_f(t),\ t \in [0, \infty) \setminus \cup_{i=1}^{\infty}[T_{i1} - 1, T_{i2} + 1]. \quad (2.37)$$

It is easy to see that u is well defined and it is an a.c. function. Equation (2.37), (2.29), and (2.28) imply that

$$\sup\{|u(t) - X_f(t)| :\ t \in [T_{i1}, T_{I2}]\} > \epsilon,\ i = 1, 2, \ldots. \quad (2.38)$$

By (2.37) and (2.36), for each integer $q \geq 1$, we have

$$I^f(0, T_{q2} + 1, u) - I^f(0, T_{q2} + 1, X_f)$$

$$= \sum_{i=1}^{q} [I^f(T_{i1}-1, T_{i2}+1, \tilde{v}_i) - I^f(T_{i1}-1, T_{i2}+1, X_f)] \leq 10 \sum_{i=1}^{q} 4^{-i} < 20.$$

In view of Proposition 2.5 u is an (f)-good function and $\lim_{t\to\infty} |X_f(t) - u(t)| = 0$. This contradicts (2.38). The contradiction we have reached completes the proof of the lemma.

2.6 Proof of Theorem 2.2

In this section we prove the next result which is an extension of Theorem 2.2.

Theorem 2.15. *Assume that $f \in \mathcal{M}$, $X_f : [0, \infty) \to R^n$ is an (f)-good function and that the properties (P1) and (P2) hold. Then for each pair of positive numbers K, ϵ there exist a pair of positive numbers δ, L and a neighborhood $\mathcal{U}$ of f in the space $\mathcal{M}$ such that the following assertion holds.*

For each $g \in \mathcal{U}$, each pair of real numbers $T_1 \geq 0$, $T_2 \geq T_1 + 2L$ and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies

$$|v(T_1)|, |v(T_2)| \leq K,\ I^g(T_1, T_2, v) \leq U^g(T_1, T_2, v(T_1), v(T_2)) + \delta,$$

the inequality $|v(t) - X_f(t)| \leq \epsilon$ holds for all $t \in [T_1 + L, T_2 - L]$.

Proof. Let $K, \epsilon > 0$ be given. Lemma 2.14 implies that there exists a pair of numbers $\delta_0 \in (0,1)$, $\tau_0 > 0$ such that the following property holds:

(C1) For each pair of numbers $T_1 \ge \tau_0$, $T_2 \ge T_1 + 1$ and each a.c. function $u : [T_1, T_2] \to R^n$ which satisfies

$$|u(T_i) - X_f(T_i)| \le \delta_0,\ i = 1, 2,\ I^f(T_1, T_2, u) \le U^f(T_1, T_2, u(T_1), u(T_2)) + \delta_0,$$

we have

$$|u(t) - X_f(t)| \le \epsilon,\ t \in [T_1, T_2].$$

Since X_f is an (f)-good function Proposition 2.6 implies that X_f is bounded. In view of Proposition 2.7 there exist

$$M_0 > K + 2 + \sup\{|X_f(t)| :\ t \in [0, \infty)\} \tag{2.39}$$

and a neighborhood $\mathcal{U}_0$ of f in $\mathcal{M}$ such that the following property holds:

(C2) For each $g \in \mathcal{U}_0$, each pair of numbers $T_1 \ge 0$, $T_2 \ge T_1 + 1$, and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies

$$|v(T_i)| \le K + 2 + \sup\{|X_f(t)| :\ t \in [0, \infty)\},\ i = 1, 2,$$
$$I^g(T_1, T_2, v) \le U^g(T_1, T_2, v(T_1), v(T_2)) + 4,$$

we have

$$|v(t)| \le M_0 \text{ for all } t \in [T_1, T_2].$$

The property (P1) implies that there exists a pair of numbers $\delta_1 \in (0, \delta_0)$, $L_1 > 0$ for which the following property holds:

(C3) For each nonnegative number T and each a.c. function $w : [T, T + L_1] \to R^n$ satisfying

$$|w(T)|, |w(T + L_1)| \le M_0 + 4,$$

$$I^f(T_1, T_2, w) \le U^f(T, T + L_1, w(T), w(T + L_1)) + \delta_1,$$

there exists a number $\tau \in [T, T + L_1]$ such that

$$|X_f(\tau) - w(\tau)| \le \delta_0.$$

By Proposition 2.8 there exists a neighborhood $\mathcal{U}_1$ of f in $\mathcal{M}$ such that the following property holds:

(C4) For each pair of numbers $T_1 \ge 0$, $T_2 \in [T_1 + L_1, T_1 + 8(L_1 + 1)]$, each function $g \in \mathcal{U}_1$ and each pair of points $x, y \in R^n$ which satisfy $|x|, |y| \le M_0 + 4$, we have

$$|U^g(T_1, T_2, x, y) - U^f(T_1, T_2, x, y)| \le \delta_1/32.$$

Proposition 2.9 implies that there exists a positive number M_1 such that

$$\sup\{|U^f(T_1, T_2, x, y)| : \ T_1 \geq 0, \ T_2 \in [T_1 + \min\{1, L_1\}, T_1 + 8(L_1 + 1)],$$
$$x, y \in R^n, \ |x, |y| \leq M_0 + 4\} \leq M_1. \tag{2.40}$$

By Proposition 2.10 there exists a neighborhood $\mathcal{U}_2$ of the function f in the space $\mathcal{M}$ such that the following property holds:

(C5) For each pair of numbers $T_1 \geq 0$, $T_2 \in [T_1 + L_1, T_1 + 8(L_1 + 1)]$, each integrand $g \in \mathcal{U}_2$, and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfy

$$\min\{I^f(T_1, T_2, v), \ I^g(T_1, T_2, v)\} \leq M_1 + 8,$$

we have $|I^f(T_1, T_2, v) - I^g(T_1, T_2, v)| \leq \delta_1/32$.

Put

$$\mathcal{U} = \mathcal{U}_0 \cap \mathcal{U}_1 \cap \mathcal{U}_2 \tag{2.41}$$

and fix a positive number

$$\delta < \min\{\epsilon, \delta_0, \delta_1\}/32 \tag{2.42}$$

and a number

$$L > 8 + 4L_1 + 4\tau_0. \tag{2.43}$$

Assume that $g \in \mathcal{U}$, $T_1 \geq 0$, $T_2 \geq T_1 + 2L$ and an a.c. function $v : [T_1, T_2] \to R^n$ satisfies

$$|v(T_i)| \leq K, \ i = 1, 2, \ I^g(T_1, T_2, v) \leq U^g(T_1, T_2, v(T_1), v(T_2)) + \delta. \tag{2.44}$$

In view of Equation (2.44) and property (C2), we have

$$|v(t)| \leq M_0 \text{ for all } t \in [T_1, T_2]. \tag{2.45}$$

Let

$$s_1, s_2 \in [T_1, T_2], \ s_2 - s_1 \in [L_1, 8(L_1 + 1)]. \tag{2.46}$$

Equation (2.45) and the property (C4) imply that

$$|U^g(s_1, s_2, v(s_1), v(s_2)) - U^f(s_1, s_2, v(s_1), v(s_2))| \leq \delta_1/32. \tag{2.47}$$

It follows from (2.40) and (2.45) that

$$U^f(s_1, s_2, v(s_1), v(s_2)) \leq M_1.$$

By this inequality and (2.47), we have

$$U^g(s_1, s_2, v(s_1), v(s_2)) \leq M_1 + \delta_1/32.$$

It follows from this inequality and (2.44) that

$$I^g(s_1, s_2, v) \le U^g(s_1, s_2, v(s_1), v(s_2)) + \delta \le M_1 + \delta_1/32 + \delta. \tag{2.48}$$

Combined with (2.46) and the property (C5), this inequality implies that

$$|I^f(s_1, s_2, v) - I^g(s_1, s_2, v)| \le \delta_1/32.$$

By this inequality, (2.48), (2.47), and (2.42), we have

$$\begin{aligned} I^f(s_1, s_2, v) &\le I^g(s_1, s_2, v) + \delta_1/32 \\ &\le U^g(s_1, s_2, v(s_1), v(s_2)) + \delta + \delta_1/32 \\ &\le U^f(s_1, s_2, v(s_1), v(s_2)) + \delta_1/32 + \delta + \delta_1/32 \end{aligned}$$

and

$$I^f(s_1, s_2, v) \le U^f(s_1, s_2, v(s_1), v(s_2)) + 3\delta_1/32. \tag{2.49}$$

Thus, we have shown that the following property holds:

(C6) Inequality (2.49) is true for each pair of numbers s_1, s_2 satisfying (2.46).

Let

$$\tau \in [T_1 + L, T_2 - L]. \tag{2.50}$$

By (2.50) and (2.43), we have

$$\tau - 1 - L_1,\ \tau + 1 + L_1 \in [T_1, T_2].$$

The property (C6) implies that

$$I^f(\tau_1 - 1 - L_1, \tau - 1, v) \le U^f(\tau_1 - 1 - L_1, \tau - 1, v(\tau_1 - 1 - L_1), v(\tau - 1)) + 3\delta_1/32, \tag{2.51}$$

$$I^f(\tau + 1, \tau + 1 + L_1, v) \le U^f(\tau + 1, \tau + 1 + L_1, v(\tau + 1), v(\tau + 1 + L_1)) + 3\delta_1/32. \tag{2.52}$$

It follows from (2.51), (2.52), (2.45), and the property (C3) that there exists a pair of numbers

$$t_1 \in [\tau - 1 - L_1, \tau - 1],\ t_2 \in [\tau + 1, \tau + 1 + L_1] \tag{2.53}$$

which satisfies

$$|X_f(t_i) - v(t_i)| \le \delta_0,\ i = 1, 2. \tag{2.54}$$

The property (C6) and (2.53) imply that

$$I^f(\tau - 1 - L_1, \tau + 1 + L_1, v)$$
$$\leq U^f(\tau - 1 - L_1, \tau + 1 + L_1, v(\tau - 1 - L_1), v(\tau + 1 + L_1)) + 3\delta_1/32,$$
$$I^f(t_1, t_2, v) \leq U^f(t_1, t_2, v(t_1), v(t_2)) + 3\delta_1/32. \tag{2.55}$$

By (2.53), (2.50), and (2.43), we have

$$t_1 \geq \tau - 1 - L_1 \geq T_1 + L - 1 - L_1 > 8 + 4L_1 + 4\tau_0 - 1 - L_1 > \tau_0.$$

It follows from this inequality, (2.55), (2.54), (2.53), the inequality $\delta_1 < \delta_0$, and the property (C1) that

$$|v(t) - X_f(t)| \leq \epsilon, \ t \in [t_1, t_2]$$

and

$$|v(\tau) - X_f(\tau)| \leq \epsilon. \tag{2.56}$$

Therefore, Inequality (2.56) holds for all $\tau \in [T_1 + L, T_2 - L]$. This completes the proof of the theorem.

Proof of Theorem 2.2: Assume that $f \in \mathcal{M}$ and that $X_1, X_2 : [0, \infty) \to R^n$ are a.c. functions which satisfy $\lim_{t\to\infty} |X_2(t) - X_1(t)| = 0$. Then f possesses the property TP, ((P1), (P2), respectively) with $X_f = X_1$, if and only if TP ((P1), (P2), respectively) holds with $X_f = X_2$.

Let $X_f : [0, \infty) \to R^n$ be a bounded a.c. function. We showed in Sect. 2.4 that if f possesses TP with the turnpike X_f, then the properties (P1) and (P2) hold.

Assume that (P1) and (P2) hold. We may assume without any loss of generality that X_f is an (f)-good function. Theorem 2.15 implies that TP holds.

2.7 Auxiliary Results for Theorem 2.4

In the sequel we use the following result [8].

Proposition 2.16. *Assume that $f \in \mathcal{M}$ and $f(t, x, \cdot) : R^n \to R^1$ is a convex function for each $(t, x) \in R^n \times [0, \infty)$. Then for each pair of numbers T_1, T_2 satisfying $0 \leq T_1 < T_2$ and each $z_1, z_2 \in R^n$, there exists an a.c. function $x : [T_1, T_2] \to R^n$ such that*

$$x(T_i) = z_i, \ i = 1, 2, \ I^f(T_1, T_2, x) = U^f(T_1, T_2, z_1, z_2).$$

Proposition 2.17. *Let $T_1 \geq 0$, $T_2 > T_1$ and let $v : [T_1, T_2] \to R^n$ be a continuous function. Assume that for each pair of real numbers $\tau_1, \tau_2 \in (T_1, T_2)$ which satisfy $\tau_1 < \tau_2$, the restriction of the function v to the interval $[\tau_1, \tau_2]$ is an a.c. function such that*

$$I^f(\tau_1, \tau_2, v) = U^f(\tau_1, \tau_2, v(\tau_1), v(\tau_2)). \tag{2.57}$$

Then the function $v : [T_1, T_2] \to R^n$ is absolutely continuous and satisfies

$$I^f(T_1, T_2, v) = U^f(T_1, T_2, v(T_1), v(T_2)). \tag{2.58}$$

Proof. Fix a number

$$M_0 > \sup\{|v(t)| : \ t \in [T_1, T_2]\}. \tag{2.59}$$

Equations (2.57) and (2.59) and Proposition 2.9 imply that the set

$$\{I^f(\tau_1, \tau_2, v) : \ \tau_1, \tau_2 \in (T_1, T_2), \ \tau_2 - \tau_1 \in (0, (T_2 - T_1)/8)\}$$

is bounded. Combined with A(ii) and Fatou's lemma this implies that the integral

$$\int_{T_1}^{T_2} f(t, v(t), v'(t))dt$$

is finite. In view of A(ii), $v' \in L^1([T_1, T_2]; R^n)$ and $v : [T_1, T_2] \to R^n$ is an a.c. function.

We claim that (2.58) is true. Let us assume the contrary. Then there exists an a.c. function $u : [T_1, T_2] \to R^n$ which satisfies

$$u(T_i) = v(T_i), \ i = 1, 2, \ I^f(T_1, T_2, v) - I^f(T_1, T_2, u) > 2\Delta \tag{2.60}$$

with $\Delta > 0$.

It is easy to see that there exists a real number $\gamma \in (0, (T_2 - T_1)/8)$ such that

$$|I^f(s_1, s_2, v)| \leq \Delta/64 \text{ for each } s_1, s_2 \in [T_1, T_1 + \gamma] \text{ satisfying } s_2 > s_1, \tag{2.61}$$
$$|I^f(s_1, s_2, v)| \leq \Delta/64 \text{ for each } s_1, s_2 \in [T_2 - \gamma, T_2] \text{ satisfying } s_2 > s_1,$$
$$|I^f(s_1, s_2, u)| \leq \Delta/64 \text{ for each } s_1, s_2 \in [T_1, T_1 + \gamma] \text{ satisfying } s_2 > s_1, \tag{2.62}$$
$$|I^f(s_1, s_2, u)| \leq \Delta/64 \text{ for each } s_1, s_2 \in [T_2 - \gamma, T_2] \text{ satisfying } s_2 > s_1.$$

Fix a real number

$$M_1 > \sup\{|v(t)| : \ t \in [T_1, T_2]\} + \sup\{|u(t)| : \ t \in [T_1, T_2]\}. \tag{2.63}$$

Proposition 2.11 implies that there exists a positive number δ such that:

For each $t_1 \geq 0$, $t_2 \in [t_1 + \gamma/16, t_1 + 16\gamma]$ and each $x_1, x_2, y_1, y_2 \in R^n$ which satisfy

$$|x_i|, |y_i| \leq M_1,\ i = 1, 2,\ |x_i - y_i| \leq \delta,\ i = 1, 2, \tag{2.64}$$

we have

$$|U^f(t_1, t_2, x_1, x_2) - U^f(t_1, t_2, y_1, y_2)| \leq \Delta/64. \tag{2.65}$$

Fix a pair of real numbers t_1, t_2 which satisfies

$$t_1 \in (T_1, T_1 + \gamma/4],\ t_2 \in [T_2 - \gamma/4, T_2], \tag{2.66}$$

$$|v(T_1) - v(t_1)|,\ |u(T_1) - u(t_1)| \leq \delta/4, \tag{2.67}$$
$$|v(T_2) - v(t_2)|,\ |u(T_2) - v(t_2)| \leq \delta/4.$$

By (2.67) and (2.60), for $i = 1, 2$, we have

$$\begin{aligned}|v(t_i) - u(t_i)| &\leq |v(t_i) - v(T_i)| + |v(T_i) - u(T_i)| \\ &+ |u(T_i) - u(t_i)| \leq \delta/2.\end{aligned} \tag{2.68}$$

There exists an a.c. function $\tilde{u} : [t_1, t_2] \to R^n$ such that

$$\begin{aligned}&\tilde{u}(t) = u(t),\ t \in [T_1 + \gamma, T_2 - \gamma],\ \tilde{u}(t_i) = v(t_i),\ i = 1, 2, \\ &I^f(t_1, T_1 + \gamma, \tilde{u}) \leq U^f(t_1, T_1 + \gamma, v(t_1), u(T_1 + \gamma)) + \Delta/128, \\ &I^f(T_2 - \gamma, t_2, \tilde{u}) \leq U^f(t_2, T_2 - \gamma, u(T_2 - \gamma), v(t_2)) + \Delta/128.\end{aligned} \tag{2.69}$$

By (2.66), the choice of γ (see (2.61), (2.62)) and (2.60), we have

$$\begin{aligned}I^f(t_1, t_2, v) - I^f(t_1, t_2, u) &= I^f(T_1, T_2, v) - I^f(T_1, T_2, u) \\ -I^f(T_1, t_1, v) - I^f(t_2, T_2, v) &+ I^f(T_1, t_1, u) + I^f(t_2, T_2, u) \\ &\geq 2\Delta - 4(\Delta/64) > 3\Delta/2.\end{aligned} \tag{2.70}$$

It follows from (2.69) and (2.57) that

$$\begin{aligned}&I^f(t_1, t_2, \tilde{u}) - I^f(t_1, t_2, u) = I^f(t_1, T_1 + \gamma, \tilde{u}) - I^f(t_1, T_1 + \gamma, u) \\ &\quad + I^f(T_2 - \gamma, t_2, \tilde{u}) - I^f(T_2 - \gamma, t_2, u) \\ &\leq U^f(t_1, T_1 + \gamma, v(t_1), u(T_1 + \gamma)) + \Delta/128 - U^f(t_1, T_1 + \gamma, u(t_1), u(T_1 + \gamma)) \\ &+ U^f(t_2, T_2 - \gamma, u(T_2 - \gamma), v(t_2)) + \Delta/128 - U^f(T_2 - \gamma, t_2, u(T_2 - \gamma), u(t_2)).\end{aligned} \tag{2.71}$$

By (2.68), (2.63), (2.66) and the choice of δ (see (2.64), (2.65)), we have

$$|U^f(t_1, T_1+\gamma, u(t_1), u(T_1+\gamma)) - U^f(t_1, T_1+\gamma, v(t_1), u(T_1+\gamma))| \le \Delta/64,$$

$$|U^f(T_2-\gamma, t_2, u(T_2-\gamma), u(t_2)) - U^f(T_2-\gamma, t_2, u(T_2-\gamma), v(t_2))| \le \Delta/64. \quad (2.72)$$

It follows from (2.71) and (2.72) that

$$I^f(t_1, t_2, \tilde{u}) - I^f(t_1, t_2, u) \le \Delta/64 + \Delta/64 + \Delta/64 < \Delta/16. \quad (2.73)$$

Equations (2.73) and (2.70) imply that

$$\begin{aligned} &I^f(t_1, t_2, v) - I^f(t_1, t_2, \tilde{u}) \\ &= I^f(t_1, t_2, v) - I^f(t_1, t_2, u) + I^f(t_1, t_2, u) - I^f(t_1, t_2, \tilde{u}) \\ &\ge 3\Delta/2 - \Delta/16 > 0, \end{aligned}$$

a contradiction (see (2.57)). The obtained contradiction completes the proof of the proposition.

2.8 Proof of Proposition 2.3

For each integrand $h \in \mathcal{M}$, each number $\delta \in (0,1)$ and each point $z \in R^n$ let an a.c. function $Z_\delta^h : [0,\infty) \to R^n$ be as guaranteed by Proposition 2.5.

Assume that $z \in R^n$, $f \in \mathcal{M}$ and that for each $(t,x) \in [0,\infty) \times R^n$, the function $f(t,x,\cdot) : R^n \to R^1$ is convex.

For each nonnegative integer i define

$$z_i^* = Z_\epsilon^f(i) \text{ with } \epsilon \in (0,1). \quad (2.74)$$

It follows from Assertion 5 of Proposition 2.5 that z_i^* ($i \ge 0$) does not depend on ϵ. Proposition 2.16 implies that there exists an a.c. function $Z^* : [0,\infty) \to R^n$ such that for each nonnegative integer i, we have

$$Z^*(i) = z_i^*, \; I^f(i, i+1, Z^*) = U^f(i, i+1, Z^*(i), Z^*(i+1)). \quad (2.75)$$

By (2.75), (2.74), and Assertion 4 of Proposition 2.5, for each natural number k and each number $\epsilon \in (0,1)$, we have

$$I^f(0,k,Z^*) = \sum_{i=0}^{k-1} I^f(i,i+1,Z^*) = \sum_{i=0}^{k-1} U^f(i,i+1,z_i^*,z_{i+1}^*)$$

$$= \sum_{i=0}^{k-1} U^f(i,i+1,Z_\epsilon^f(i),Z_\epsilon^f(i+1)) \le I^f(0,k,Z_\epsilon^f)$$

$$\le U^f(0,k,Z_\epsilon^f(0),Z_\epsilon^f(k)) + \epsilon = U^f(0,k,Z^*(0),Z^*(k)) + \epsilon.$$

Since ϵ is an arbitrary element of $(0,1)$ we conclude that for any nonnegative integer k,

$$I^f(0,k,Z^*) = U^f(0,k,Z^*(0),Z^*(k)).$$

This implies that $I^f(0,T,Z^*) = U^f(0,T,Z^*(0),Z^*(T))$ for any $T > 0$. In view of Assertion 1 of Proposition 2.5 and Proposition 2.1, the function Z^* is bounded and (f)-good. This completes the proof of Proposition 2.3.

2.9 Overtaking Optimal Trajectories

Proposition 2.18. *Let $f \in \mathcal{M}$, property (Q1) hold (see Theorem 2.4) and let $x : [0,\infty) \to R^n$ be a bounded a.c. function such that for each positive number T,*

$$I^f(0,T,x) = U^f(0,T,x(0),x(T)). \tag{2.76}$$

Then x is an (f)-overtaking optimal function.

Proof. Proposition 2.1 and (2.76) imply that x is an (f)-good function. Assume that the function x is not (f)-overtaking optimal. Then there exists an a.c. function $y : [0,\infty) \to R^n$ which satisfies

$$y(0) = x(0), \ \limsup_{T\to\infty}[I^f(0,T,x) - I^f(0,T,y)] \ge 2\epsilon \tag{2.77}$$

with some positive number ϵ. It follows from Proposition 2.5 that there exists a bounded (f)-good function $Z : [0,\infty) \to R^n$ satisfying $Z(0) = x(0)$ and such that for each a.c. function $v : [0,\infty) \to R^n$, either

$$\lim_{T\to\infty}[I^f(0,T,v) - I^f(0,T,Z)] = \infty \tag{2.78}$$

or

$$\sup\{|I^f(0,T,v) - I^f(0,T,Z)| : \ T \in (0,\infty)\} < \infty, \tag{2.79}$$

$$\sup\{|v(t)| : \ t \in [0,\infty)\} < \infty.$$

Since the function x is (f)-good, we have

$$\sup\{|I^f(0,T,x) - I^f(0,T,Z)| : \ T \in (0,\infty)\} < \infty. \tag{2.80}$$

In view of (2.77) and (2.80), equation (2.78) does not hold with $v = y$. Hence (2.79) is valid with $v = y$. This implies that y is a bounded (f)-good function. By the property (Q1), we have

$$\lim_{t\to\infty} |x(t) - y(t)| = 0. \tag{2.81}$$

Since x, y are bounded functions there exists a real number

$$\Delta > \sup\{|x(t)| + |y(t)| : \ t \in [0,\infty)\} + 2. \tag{2.82}$$

By Proposition 2.11 there is a positive number δ such that for each $T \geq 0$ and each $z_i \in R^n$, $i = 1,2,3,4$ which satisfy

$$|z_i| \leq \Delta, \ i = 1,2,3,4, \ |z_1 - z_3|, |z_2 - z_4| \leq \delta,$$

we have

$$|U^f(T, T+1, z_1, z_2) - U^f(T, T+1, z_3, z_4)| \leq \epsilon/8. \tag{2.83}$$

By (2.77) there exists a sequence $\{T_i\}_{i=1}^{\infty} \subset (0,\infty)$ which satisfies

$$T_{i+1} \geq T_i + 8, \ i = 1,2,\ldots, \ I^f(0,T_i,x) - I^f(0,T_i,y) > 3\epsilon/2, \ i = 1,2,\ldots. \tag{2.84}$$

It follows from (2.81) that there is an integer $j \geq 1$ such that

$$|x(T_j) - y(T_j)| \leq \delta. \tag{2.85}$$

Consider an a.c. function $\tilde{x} : [0, T_j + 1] \to R^n$ which satisfies

$$\tilde{x}(t) = y(t), \ t \in [0, T_j], \ \tilde{x}(T_{j+1}) = x(T_j + 1),$$

$$I^f(T_j, T_j + 1, \tilde{x}) \leq U^f(T_j, T_j + 1, y(T_j), x(T_j + 1)) + \epsilon/8. \tag{2.86}$$

By (2.77) and (2.86), we have

$$\tilde{x}(0) = x(0), \ \tilde{x}(T_j + 1) = x(T_j + 1). \tag{2.87}$$

In view of (2.86) and (2.84),

$$\begin{aligned}&I^f(0, T_J + 1, \tilde{x}) - I^f(0, T_j + 1, x)\\
&= I^f(0, T_j, \tilde{x}) + I^f(T_j, T_j + 1, \tilde{x}) - I^f(0, T_j, x) - I^f(T_j, T_j + 1, x)\\
&\le I^f(0, T_j, y) - I^f(0, T_j, x) +\\
&U^f(T_j, T_j + 1, y(T_j), x(T_j + 1)) + \epsilon/8\\
&-U^f(T_j, T_j + 1, x(T_j), x(T_j + 1)) < -3\epsilon/2 + \epsilon/8\\
&+ U^f(T_j, T_j + 1, y(T_j), x(T_j + 1)) - U^f(T_j, T_j + 1, x(T_j), x(T_j + 1)).\end{aligned} \tag{2.88}$$

It follows from (2.82), (2.85), and the choice of δ (see (2.83)) that

$$|U^f(T_j, T_j + 1, y(T_j), x(T_j + 1)) - U^f(T_j, T_j + 1, x(T_j), x(T_j + 1))| \le \epsilon/8.$$

By this inequality and (2.88),

$$I^f(0, T_j + 1, \tilde{x}) - I^f(0, T_j + 1, x) < -3\epsilon/2 + \epsilon/8 + \epsilon/8 < 0.$$

This contradicts (2.76). The contradiction we have reached completes the proof of the proposition.

Propositions 2.3 and 2.18 imply the following result.

Proposition 2.19. *Let $f \in \mathcal{M}$, for each $(t, x) \in [0, \infty) \times R^n$, the function $f(t, x, \cdot) : R^n \to R^1$ be convex, and let property (Q1) hold (see Theorem 2.4). Then for each $z \in R^n$ there exists a bounded (f)-overtaking optimal function $Z : [0, \infty) \to R^n$ such that $Z(0) = z$.*

Proposition 2.20. *Let $f \in \mathcal{M}$ and property (Q1) hold (see Theorem 2.4). Assume that $v_1, v_2 : [0, \infty) \to R^n$ are bounded a.c. functions, v_1 is (f)-overtaking optimal, $T_0 > 0$,*

$$v_1(t) = v_2(t),\ t \in [0, T_0] \tag{2.89}$$

and that

$$I^f(T_0, \tau, v_2) = U^f(T_0, \tau, v_2(T_0), v_2(\tau)) \text{ for each } \tau > T_0. \tag{2.90}$$

Then v_2 is an (f)-overtaking optimal function.

Proof. It is clear that v_1 is an (f)-good function. We claim that v_2 is an (f)-good function. Fix a real number

$$M_0 > \sup\{|v_i(t)| :\ t \in [0, \infty),\ i = 1, 2\}. \tag{2.91}$$

In view of Proposition 2.9 there exists a positive number M_1 which satisfies

$$M_1 > \sup\{|U^f(t_1, t_2, y, z)| : \ t_1 \geq 0, \ t_2 \in [t_1 + 1/8, t_1 + 8],$$
$$y, z \in R^n, \ |y|, |z| \leq M_0 + 2\}. \tag{2.92}$$

Let $\tau \geq T_0 + 2$ be given. Consider an a.c. function $u : [0, \infty) \to R^n$ which satisfies

$$u(t) = v_1(t), \ t \in [0, \tau - 1], \ u(\tau) = v_2(\tau),$$
$$I^f(\tau - 1, \tau, u) \leq U^f(\tau - 1, \tau, v_1(\tau - 1), v_2(\tau)) + 1. \tag{2.93}$$

By (2.93) and (2.89), we have

$$u(T_0) = v_2(T_0), \ u(\tau) = v_2(\tau). \tag{2.94}$$

It follows from (2.90), (2.89), (2.93), and (2.94) that

$$I^f(0, \tau, u) - I^f(0, \tau, v_2) = I^f(T_0, \tau, u) - I^f(T_0, \tau, v_2) \geq 0. \tag{2.95}$$

Relations (2.93), (2.91), and (2.92) imply that

$$I^f(0, \tau, u) - I^f(0, \tau, v_1) = I^f(\tau - 1, \tau, u) - I^f(\tau - 1, \tau, v_1)$$
$$\leq U^f(\tau - 1, \tau, v_1(\tau - 1), v_2(\tau)) + 1 - U^f(\tau - 1, \tau, v_1(\tau - 1), v_1(\tau)) \leq 2M_1.$$

In view of this relation and (2.95),

$$I^f(0, \tau, v_2) \leq I^f(0, \tau, u) \leq I^f(0, \tau, v_1) + 2M_1$$

and

$$I^f(0, \tau, v_2) \leq I^f(0, \tau, v_1) + 2M_1 \text{ for all } \tau > T_0. + 2.$$

Since v_1 is (f)-good we conclude that v_2 is an (f)-good function. The property (Q1) implies that

$$\lim_{t \to \infty} |v_2(t) - v_1(t)| = 0. \tag{2.96}$$

Since v_1 is (f)-overtaking optimal we conclude that

$$\limsup_{T \to \infty} [I^f(0, T, v_1) - I^f(0, T, v_2)] \leq 0. \tag{2.97}$$

We claim that

$$\limsup_{T \to \infty} [I^f(0, T, v_2) - I^f(0, T, v_1)] \leq 0. \tag{2.98}$$

Let $\epsilon > 0$ be given. In view of Proposition 2.11 there exists a positive number δ such that for each nonnegative number t and each $y_i, z_i \in R^n$, $i = 1, 2$ which satisfy

$$|y_i|, |z_i| \leq M_0 + 1,\ i = 1, 2,\ |y_i - z_i| \leq \delta,\ i = 1, 2, \tag{2.99}$$

we have

$$|U^f(t, t+1, y_1, y_2) - U^f(t, t+1, z_1, z_2)| \leq \epsilon/8. \tag{2.100}$$

By (2.96) there exists a real number $T_1 > T_0 + 4$ for which

$$|v_2(t) - v_1(t)| \leq \delta \text{ for all } t \in [T_1, \infty). \tag{2.101}$$

Let $T > T_1$ be given and consider an a.c. function $w : [0, T+1] \to R^n$ which satisfies

$$w(t) = v_1(t),\ t \in [0, T],\ w(T+1) = v_2(T+1),$$
$$I^f(T, T+1, w) \leq U^f(T, T+1, v_1(T), v_2(T+1)) + \epsilon/8. \tag{2.102}$$

By (2.102) and (2.89),

$$w(t) = v_2(t),\ t \in [0, T_0],\ w(T+1) = v_2(T+1). \tag{2.103}$$

It follows from (2.103) and (2.90) that

$$I^f(0, T+1, w) - I^f(0, T+1, v_2) =$$
$$I^f(T_0, T+1, w) - I^f(T_0, T+1, v_2) \geq 0. \tag{2.104}$$

In view of (2.102), (2.91), (2.101), and the choice of δ (see (2.99), (2.100)), we have

$$I^f(0, T+1, w) - I^f(0, T+1, v_1) = I^f(T, T+1, w) - I^f(T, T+1, v_1)$$
$$\leq U^f(T, T+1, v_1(T), v_2(T+1)) + \epsilon/8$$
$$- U^f(T, T+1, v_1(T), v_1(T+1)) \leq \epsilon/8 + \epsilon/8 = \epsilon/4. \tag{2.105}$$

Relations (2.105) and (2.104) imply that

$$I^f(0, T+1, v_2) \leq I^f(0, T+1, w) \leq I^f(0, T+1, v_1) + \epsilon/4$$

for all $T > T_1$. This implies (2.98). It follows from (2.98) and (2.97) that

$$\lim_{T \to \infty} [I^f(0, T, v_1) - I^f(0, T, v_2)] = 0.$$

Since v_1 is (f)-overtaking optimal we conclude that v_2 is (f)-overtaking optimal. This completes the proof of the proposition.

2.10 STP Implies (Q1), (Q2), and (Q3)

Let $f \in \mathcal{M}$, for each $(t,x) \in [0,\infty) \times R^n$, the function $f(t,x,\cdot) : R^n \to R^1$ be convex, and the function f possess STP, and let a bounded a.c. function $X_f : [0,\infty) \to R^n$ be the turnpike of f. In Sect. 2.4 we showed that properties (Q1) and (Q3) hold. Now we claim that (Q2) holds.

Proposition 2.19 implies that there is an (f)-overtaking optimal function $v : [0,\infty) \to R^n$ satisfying $v(0) = X_f(0)$. Assume that $v : [0,\infty) \to R^n$ is an (f)-overtaking optimal function for which $v(0) = X_f(0)$. By Proposition 2.5, v is bounded. Then STP implies that $v(t) = X_f(t)$ for all $t \in [0,\infty)$.

2.11 The Basic Lemma for Theorem 2.4

Assume that $f \in \mathcal{M}$, for each $(t,x) \in [0,\infty) \times R^n$, the function $f(t,x,\cdot) : R^n \to R^1$ is convex, and $X_f : [0,\infty) \to R^n$ is a bounded a.c. function, and assume that properties (Q1), (Q2), and (Q3) hold.

Lemma 2.21 (Basic Lemma.). *Let ϵ be a positive number. Then there exists a positive number δ such that for each pair of numbers $T_1 \geq 0$, $T_2 \geq T_1 + 1$ and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies*

$$|v(T_i) - X_f(T_i)| \leq \delta,\ i = 1,2,\ I^f(T_1,T_2,v) \leq U^f(T_1,T_2,v(T_1),v(T_2)) + \delta, \tag{2.106}$$

the following inequality holds:

$$|X_f(t) - v(t)| \leq \epsilon \text{ for all } t \in [T_1,T_2]. \tag{2.107}$$

Proof. Lemma 2.14 implies that there exists a pair of numbers $\tau_0, \delta_0 \in (0, \epsilon/16)$ for which the following property holds:

(Q4) For each pair of numbers $T_1 \geq \tau_0$, $T_2 \geq T_1 + 1$ and each a.c. function $v : [T_1,T_2] \to R^n$ which satisfies

$$|v(T_i) - X_f(T_i)| \leq \delta_0,\ i = 1,2,\ I^f(T_1,T_2,v) \leq U^f(T_1,T_2,v(T_1),v(T_2)) + \delta_0,$$

we have

$$|v(t) - X_f(t)| \leq \epsilon,\ t \in [T_1,T_2].$$

We may assume without any loss of generality that $\delta_0 < 1$. Fix a number

$$M_0 > 4 + \sup\{|X_f(t)| :\ t \in [0,\infty)\}. \tag{2.108}$$

In view of Proposition 2.7 there is a real number $M_1 > 1$ such that for each pair of numbers $T_1 \geq 0$, $T_2 \geq T_1 + 8^{-1}$ and each a.c. function $v : [T_1,T_2] \to R^n$ which satisfies

$$|v(T_i)| \leq M_0 + 4,\ i = 1,2,\ I^f(T_1,T_2,v) \leq U^f(T_1,T_2,v(T_1),v(T_2)) + 4, \tag{2.109}$$

we have

$$|v(t)| \le M_1, \ t \in [T_1, T_2]. \tag{2.110}$$

The property (Q3) implies that there exist

$$\delta_1 \in (0, \min\{1, \delta_0\})$$

and a positive number L_1 such that the following property holds:

(Q5) For each nonnegative number T and each a.c. function $v : [T, T + L_1] \to R^n$ satisfying

$$|v(T)|, |v(T + L_1)| \le M_1 + 4,$$

$$I^f(T, T + L_1, v) \le U^f(T, T + L_1, v(T), v(T + L_1)) + \delta_1, \tag{2.111}$$

there exists a number $\tau \in [T, T + L_1]$ such that

$$|X_f(\tau) - v(\tau)| \le \delta_0. \tag{2.112}$$

Consider a sequence $\{\delta_i\}_{i=1}^{\infty} \subset (0, 1)$ for which

$$\delta_i < 2^{-1}\delta_{i-1}, \ i = 2, 3, \dots. \tag{2.113}$$

Assume that the lemma does not hold. Then for each integer $i \ge 1$, there exist $T_{i1} \ge 0$, $T_{i2} \ge T_{i1} + 1$, an a.c. function $v_i : [T_{i1}, T_{i2}] \to R^n$ which satisfies

$$|X_f(T_{ij}) - v_i(T_{ij})| \le \delta_i, \ j = 1, 2, \ I^f(T_{i1}, T_{i2}, v_i) \le$$

$$U^f(T_{i1}, T_{i2}, v_i(T_{i1}), v_i(T_{i2})) + \delta_i, \tag{2.114}$$

and a number $t_i \in [T_{i1}, T_{i2}]$ such that

$$|X_f(t_i) - v_i(t_i)| > \epsilon. \tag{2.115}$$

Let i be a natural number. By the property (Q4), (2.114), (2.115), (2.113), and (2.110), we have

$$T_{i1} < \tau_0. \tag{2.116}$$

It follows from (2.114), (2.113), (2.108), and the choice of M_1 (see (2.109), (2.110)) that

$$|v_i(t)| \le M_1 \text{ for all } t \in [T_{i1}, T_{i2}]. \tag{2.117}$$

We claim that $t_i \le \tau_0 + L_1 + 2$. Let us assume the contrary. Then

$$t_i > \tau_0 + L_1 + 2. \tag{2.118}$$

Consider the restriction of v_i to the interval

$$[t_i - L_1 - 1, t_i - 1] \subset (\tau_0 + 1, \infty). \tag{2.119}$$

In view of the property (Q5), (2.119), (2.117), (2.114), and (2.113), there exists a number

$$\hat{t} \in [t_i - L_1 - 1, t_i - 1] \tag{2.120}$$

for which

$$|X_f(\hat{t}) - v_i(\hat{t})| \le \delta_0. \tag{2.121}$$

It follows from (2.120) and (2.118) that

$$\hat{t} > \tau_0 + 1,\ T_{i2} - \hat{t} \ge 1.$$

By these inequalities, (2.121), (2.114), (2.113), and property (Q4), we have

$$|v_i(t) - X_f(t)| \le \epsilon \text{ for all } t \in [\hat{t}, T_{i2}].$$

It follows from this inequality and (2.120) that

$$|v_i(t_i) - X_f(t_i)| \le \epsilon,$$

a contradiction. The contradiction we have reached proves that

$$t_i \le \tau_0 + L_1 + 2. \tag{2.122}$$

Extracting if it is necessary a subsequence and re-indexing, we may assume without any loss of generality that there exist

$$\tilde{T}_1 = \lim_{i\to\infty} T_{i1} \in [0, \tau_0],\ \tilde{t} = \lim_{i\to\infty} t_i \in [\tilde{T}_1, \tau_0 + L_1 + 2],$$

$$\tilde{T}_2 = \lim_{i\to\infty} T_{i2} \in [\tilde{t}, \infty] \tag{2.123}$$

(see (2.116), (2.122)).

In view of A(ii), (2.114), (2.117), and Proposition 2.9, for each $\tau_1 \in (\tilde{T}_1, \tilde{T}_2)$, $\tau_2 \in (\tau_1, \tilde{T}_2)$ the sequence $\{I^f(\tau_1, \tau_2, v_i)\}_{i=1}^{\infty}$ is bounded.

By lower semicontinuity results [8] we may assume that there exists a function $\hat{v} : (\tilde{T}_1, \tilde{T}_2) \to R^n$ such that the following property holds:

(Q6) For each number $\tau_1 \in (\tilde{T}_1, \tilde{T}_2)$ and each number $\tau_2 \in (\tau_1, \tilde{T}_2)$, the function $\hat{v}$ is a.c. on $[\tau_1, \tau_2]$,

$$v_i(t) \to \hat{v}(t) \text{ as } i \to \infty \text{ uniformly in } t \in [\tau_1, \tau_2],$$

$$v_i' \to \hat{v}' \text{ as } i \to \infty \text{ weakly in } L^1([\tau_1, \tau_2]; R^n)$$

and

$$I^f(\tau_1, \tau_2, \hat{v}) \le \liminf_{i\to\infty} I^f(\tau_1, \tau_2, v_i).$$

We claim that

$$X_f(\tilde{T}_1) = \lim_{t\to\tilde{T}_1^+} \hat{v}(t).$$

Let $\Delta > 0$ be given. It follows from Propositions 2.12 and 2.9, (2.114) and (2.117) that there exists a number $\gamma \in (0, 1/8)$ such that the following properties hold:

For each natural number i and each pair of numbers $t_1, t_2 \in [T_{i1}, T_{i2}]$ which satisfy $|t_1 - t_2| \le 4\gamma$, we have

$$|v_i(t_1) - v_i(t_2)| \le \Delta; \tag{2.124}$$

for each pair of numbers $t_1, t_2 \in [0, \infty)$ which satisfy $|t_1 - t_2| \le 4\gamma$, we have

$$|X_f(t_1) - X_f(t_2)| \le \Delta. \tag{2.125}$$

Let $\tau \in (\tilde{T}_1, \tilde{T}_1 + \gamma)$ be given. Then for all sufficiently large integers $i \ge 1$, the inequalities

$$T_{i1} < \tau < \tilde{T}_1 + \gamma < T_{i1} + 2\gamma \tag{2.126}$$

hold, and by the choice of γ, we have

$$|v_i(\tau) - v_i(T_{i1})| \le \Delta. \tag{2.127}$$

By (2.126), (2.127), and (2.114), for all sufficiently large integers $i \ge 1$,

$$|v_i(\tau) - X_f(T_{i1})| \le |v_i(\tau) - v_i(T_{i1})| +$$

$$|v_i(T_{i1}) - X_f(T_{i1})| \le \Delta + \delta_i. \tag{2.128}$$

It follows from the choice of γ, (2.125), and (2.123) that for all sufficiently large natural numbers i, we have

$$|X_f(\tilde{T}_1) - X_f(T_{i1})| \le \Delta.$$

By this inequality and (2.128), for all sufficiently large natural numbers i,

$$|v_i(\tau) - X_f(\tilde{T}_1)| \le |v_i(\tau) - X_f(T_{i1}|$$

$$+|X_f(T_{i1}) - X_f(\tilde{T}_1)| \le \Delta + \delta_i + \Delta.$$

Hence (Q6) and (2.113) imply that

$$|\hat{v}(\tau) - X_f(\tilde{T}_1)| = \lim_{i\to\infty} |v_i(\tau) - X_f(\tilde{T}_1)| \le \lim_{i\to\infty} 2\Delta + \delta_i = 2\Delta.$$

We have shown that for each $\tau \in (\tilde{T}_1, \tilde{T}_1 + \gamma)$, the inequality

$$|\hat{v}(\tau) - X_f(\tilde{T}_1)| \le 2\Delta$$

is true. Since Δ is an arbitrary positive number we conclude that

$$X_f(\tilde{T}_1) = \lim_{\tau\to\tilde{T}_1^+} \hat{v}(\tau). \tag{2.129}$$

Analogously it is shown that if $\tilde{T}_2 < \infty$, then

$$X_f(\tilde{T}_2) = \lim_{\tau \to \tilde{T}_2^-} \hat{v}(t). \tag{2.130}$$

Define

$$\hat{v}(\tilde{T}_1) = X_f(\tilde{T}_1), \text{ and if } \tilde{T}_2 < \infty, \text{ then } \hat{v}(\tilde{T}_2) = X_f(\tilde{T}_2).$$

By the property (Q6), (2.114), (2.113), and Proposition 2.11, for each pair of numbers $S_1, S_2 \in (\tilde{T}_1, \tilde{T}_2)$ which satisfy $S_1 < S_2$, we have

$$I^f(S_1, S_2, \hat{v}) \leq \liminf_{i \to \infty} I^f(S_1, S_2, v_i)$$

$$\leq \liminf_{i \to \infty} [U^f(S_1, S_2, v_i(S_1), v_i(S_2)) + \delta_i]$$

$$= \liminf_{i \to \infty} U^f(S_1, S_2, v_i(S_1), v_i(S_2)) = U^f(S_1, S_2, \hat{v}(S_1), \hat{v}(S_2))$$

and

$$I^f(S_1, S_2, \hat{v}) = U^f(S_1, S_2, \hat{v}(S_1), \hat{v}(S_2)). \tag{2.131}$$

It follows from Property (Q6), (2.117), (2.129), and (2.130) that $\hat{v}$ is bounded. By Proposition 2.17, Property (Q6), (2.29), (2.131), and (2.130), $\hat{v}$ is a.c. function on $[\tilde{T}_1, \tau)$ for each real number $\tau \leq \tilde{T}_2$, and the following properties hold:

$$I^f(\tilde{T}_1, \tau, \hat{v}) = U^f(\tilde{T}_1, \tau, \hat{v}(T_1), \hat{v}(\tau)) \tag{2.132}$$

for each $\tau \in (\tilde{T}_1, \tilde{T}_2]$ if $\tilde{T}_2 < \infty$; equality (2.132) is valid for each real number $\tau > \tilde{T}_1$ if $\tilde{T}_2 = \infty$.

We claim that $\hat{v}(\tilde{t}) \neq X_f(\tilde{t})$. By Proposition 2.9, there exists a positive number M_2 for which

$$\sup\{|U^f(s_1, s_2, x, y)| : \; s_1 \geq 0, \; s_2 \in [s_1 + 8^{-1}, s_1 + 8],$$

$$x, y \in R^n, \; |x|, |y| \leq M_1 + 2\} + 4 < M_2. \tag{2.133}$$

It follows from (2.133), (2.117), and (2.114) that the following property holds:

(Q7) For each natural number i and each pair of numbers $s_1, s_2 \in [T_{i1}, T_{i2}]$ which satisfies $s_2 \in [s_1 + 8^{-1}, s_1 + 8]$, we have

$$I^f(s_1, s_2, v_i) < M_2.$$

By the property (Q2), Proposition 2.9, and (2.108),

$$\sup\{I^f(s_1, s_2, X_f) : \; s_1 \geq 0, \; s_2 \in [s_1 + 8^{-1}, s_1 + 8]\} < \infty. \tag{2.134}$$

It follows from the property (Q7), (2.134), and Proposition 2.12 that there exists a positive number

$$\gamma < \min\{1, \tilde{T}_2 - \tilde{T}_1\}/32$$

for which the following properties hold:

(Q8) For each pair of numbers $s_1, s_2 \geq 0$ satisfying $|s_1 - s_2| \leq \gamma$, we have

$$|X_f(s_1) - X_f(s_2)| \leq \epsilon/64.$$

(Q9) For each natural number i and each pair of numbers $s_1, s_2 \in [T_{i1}, T_{i2}]$ satisfying $|s_1 - s_2| \leq \gamma$, we have

$$|v_i(s_1) - v_i(s_2)| \leq \epsilon/64.$$

Let $i \geq 1$ be an integer. We claim that $t_i - T_{i1} > \gamma$. Let us assume the contrary. Then $t_i - T_{i1} \leq \gamma$ and properties (Q8) and (Q9) imply that

$$|X_f(t_i) - X_f(T_{i1})|,\ |v_i(t_i) - v_i(T_{i1})| \leq \epsilon/64.$$

By these inequalities and (2.114), we have

$$|X_f(t_i) - v_i(t_i)| \leq |X_f(t_i) - X_f(T_{i1})| +$$

$$|X_f(T_{i1}) - v_i(T_{i1})| + |v_i(T_{i1}) - v_i(t_i)| \leq \epsilon/64 + \delta_i + \epsilon/64$$

$$\leq \epsilon/32 + \delta_0 < \epsilon/32 + \epsilon/16 < \epsilon/2,$$

$$|X_f(t_i) - v_i(t_i)| < \epsilon/2.$$

The inequality above contradicts (2.115). The obtained contradiction proves that

$$t_i - T_{i1} > \gamma, \tag{2.135}$$

as claimed. Analogously it is shown that

$$T_{i2} - t_i > \gamma. \tag{2.136}$$

By (2.135), (2.136), (2.122), (2.123), and property (Q6),

$$\lim_{i \to \infty} |v_i(t_i) - \hat{v}(t_i)| = 0.$$

In view of this equality and (2.123), we have

$$\lim_{i \to \infty} v_i(t_i) = \hat{v}(\tilde{t}). \tag{2.137}$$

Relation (2.123) implies that $\lim_{i\to\infty} X_f(t_i) = X_f(\tilde{t})$. By the equality above, (2.137), and (2.115),

$$|\hat{v}(\tilde{t}) - X_f(\tilde{t})| = \lim_{i\to\infty} |v_i(t_i) - X_f(t_i)| \geq \epsilon. \tag{2.138}$$

Hence,

$$\hat{v}(\tilde{t}) \neq X_f(\tilde{t}). \tag{2.139}$$

There are two cases: (1) $\tilde{T}_2 = \infty$; (2) $\tilde{T}_2 < \infty$. Let $\tilde{T}_2 < \infty$. Then Equality (2.132) is true for each $\tau \in (\tilde{T}_1, \tilde{T}_2)$. Proposition 2.17 implies that $\hat{v} : [\tilde{T}_1, \tilde{T}_2] \to R^n$ is an a.c. function and that

$$I^f(\tilde{T}_1, \tilde{T}_2, \hat{v}) = U^f(\tilde{T}_1, \tilde{T}_2, \hat{v}(\tilde{T}_1), \hat{v}(\tilde{T}_2)). \tag{2.140}$$

Clearly,

$$X_f(\tilde{T}_i) = \hat{v}(\tilde{T}_i),\ i = 1, 2.$$

Define an a.c. function $u : [0,\infty) \to R^n$ as follows:

$$u(t) = X_f(t),\ t \in [0,\infty) \setminus (\tilde{T}_1, \tilde{T}_2),$$

$$u(t) = \hat{v}(t),\ t \in (T_1, T_2).$$

Evidently, the function u is well defined. It follows from the property (Q2) and (2.140) that the function $\hat{v}$ is (f)-overtaking optimal. On the other hand, $u(0) = X_f(0)$ and $u(\tilde{t}) = \hat{v}(\tilde{t}) \neq X_f(\tilde{t})$. This contradicts property (Q2). Hence, the case (2) does not hold and $\tilde{T}_2 = \infty$.

For each nonnegative number t satisfying $t < \tilde{T}_1$, put $\hat{v}(t) = X_f(t)$. Now equality (2.132) is true for each $\tau > \tilde{T}_1$. Combined with the boundedness of $\hat{v}$, the equality $\hat{v}(T_1) = X_f(\tilde{T}_1)$, and Proposition 2.20, this implies that $\hat{v}$ is (f)-overtaking optimal. Now (2.139) contradicts property (Q2). The obtained contradiction completes the proof of the lemma.

2.12 Proof of Theorem 2.4

In this section we prove the next theorem which is a generalization of Theorem 2.4.

Theorem 2.22. *Let $f \in \mathcal{M}$, for each $(t,x) \in [0,\infty) \times R^n$ the function $f(t,x,\cdot) : R^n \to R^1$ be convex, and let $X_f : [0,\infty) \to R^n$ be a bounded a.c. function. Assume that the properties (Q1), (Q2), and (Q3) from Theorem 2.4 hold.*

Then for each pair of positive numbers K, ϵ, there exist a pair of positive numbers δ, L and a neighborhood $\mathcal{U}$ of the integrand f in the space $\mathcal{M}$ such that the following property holds:

For each integrand $g \in \mathcal{U}$, each pair of real numbers $T_1 \geq 0$, $T_2 \geq T_1 + 2L$, and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies

$$|v(T_1)|, |v(T_2)| \leq K, \; I^g(T_1, T_2, v) \leq U^g(T_1, T_2, v(T_1), v(T_2)) + \delta, \quad (2.141)$$

there exists a pair of real numbers $\tau_1 \in [T_1, T_1 + L]$, $\tau_2 \in [T_2 - L, T_2]$ such that

$$|v(t) - X_f(t)| \leq \epsilon \text{ for all } t \in [\tau_1, \tau_2].$$

Moreover, if $|v(T_1) - X_f(T_1)| \leq \delta$, then $\tau_1 = T_1$, and if $|v(T_2) - X_f(T_2)| \leq \delta$, then $T_2 = \tau_2$.

Proof. Let $K, \epsilon > 0$ be given. In view of Lemma 2.21, there exists a real number $\delta_0 \in (0, 1)$ for which the following property holds:

(C1) For each pair of real numbers $T_1 \geq 0$, $T_2 \geq T_1 + 1$ and each an a.c. function $v : [T_1, T_2] \to R^n$ which satisfies

$$|v(T_i) - X_f(T_i)| \leq \delta_0, \; i = 1, 2, \; I^f(T_1, T_2, v) \leq U^f(T_1, T_2, v(T_1), v(T_2)) + \delta_0,$$

we have

$$|v(t) - X_f(t)| \leq \epsilon \text{ for all } t \in [T_1, T_2].$$

Proposition 2.7 implies that there exist a real number

$$M_0 > K + 2 + \sup\{|X_f(t)| : \; t \in [0, \infty)\} \quad (2.142)$$

and a neighborhood $\mathcal{U}_0$ of the integrand f in the space $\mathcal{M}$ such that the following property holds:

(C2) For each integrand $g \in \mathcal{U}_0$, each pair of real numbers $T_1 \geq 0$, $T_2 \geq T_1 + 1$, and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies

$$|v(T_i)| \leq K + 2 + \sup\{|X_f(t)| : \; t \in [0, \infty)\}, \; i = 1, 2,$$

$$I^g(T_1, T_2, v) \leq U^g(T_1, T_2, v(T_1), v(T_2)) + 4,$$

we have

$$|v(t)| \leq M_0 \text{ for all } t \in [T_1, T_2].$$

By the property (Q3) there exists a pair of real numbers $\delta_1 \in (0, \delta_0)$, $L_1 > 0$ for which the following property holds:

(C3) For each nonnegative number T and each a.c. function $w : [T, T + L_1] \to R^n$ satisfying

$$|w(T)|, |w(T + L_1)| \leq M_0 + 4,$$

$$I^f(T, T + L_1, w) \leq U^f(T, T + L_1, w(T), w(T + L_1)) + \delta_1,$$

there exists a real number $\tau \in [T, T + L_1]$ such that $|X_f(\tau) - w(\tau)| \le \delta_0$.

By Proposition 2.8, there exists a neighborhood $\mathcal{U}_1$ of the integrand f in the space $\mathcal{M}$ for which the following property holds:

(C4) For each pair of real numbers $T_1 \ge 0$, $T_2 \in [T_1 + L_1, T_1 + 8(L_1 + 1)]$, each integrand $g \in \mathcal{U}_1$, and each pair of points $x, y \in R^n$ satisfying $|x|, |y| \le M_0 + 4$, we have

$$|U^g(T_1, T_2, x, y) - U^f(T_1, T_2, x, y)| \le \delta_1/32.$$

In view of Proposition 2.9, there exists a positive number M_1 such that

$$\sup\{|U^f(T_1, T_2, x, y)| : \ T_1 \ge 0, \ T_2 \in [T_1 + 1, T_1 + 8(L_1 + 1)],$$
$$x, y \in R^n, \ |x|, |y| \le M_0 + 4\} \le M_1. \tag{2.143}$$

By Proposition 2.10, there exists a neighborhood $\mathcal{U}_2$ of the integrand f in the space $\mathcal{M}$ such that the following property holds:

(C5) For each pair of real numbers $T_1 \ge 0$, $T_2 \in [T_1 + L_1, T_1 + 8(L_1 + 1)]$, each integrand $g \in \mathcal{U}_2$ and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies

$$\min\{I^f(T_1, T_2, v), \ I^g(T_1, T_2, v)\} \le M_1 + 8,$$

we have

$$|I^f(T_1, T_2, v) - I^g(T_1, T_2, v)| \le \delta_1/32.$$

Define

$$\mathcal{U} = \mathcal{U}_0 \cap \mathcal{U}_1 \cap \mathcal{U}_2, \tag{2.144}$$

fix a positive number

$$\delta < \min\{\epsilon, \delta_0, \delta_1\}/32,$$

and put

$$L = 8 + 6L_1. \tag{2.145}$$

Assume that $g \in \mathcal{U}$, $T_1 \ge 0$, $T_2 \ge T_1 + 2L$ and that an a.c. function $v : [T_1, T_2] \to R^n$ satisfies

$$|v(T_i)| \le K, \ i = 1, 2, \ I^g(T_1, T_2, v) \le U^g(T_1, T_2, v(T_1), v(T_2)) + \delta. \tag{2.146}$$

In view of (2.146), (2.144), and property (C2), we have

$$|v(t)| \le M_0 \text{ for all } t \in [T_1, T_2]. \tag{2.147}$$

Let

$$s_1, s_2 \in [T_1, T_2] \text{ and } s_2 - s_1 \in [L_1, 8(L_1 + 1)]. \tag{2.148}$$

By (2.147), (2.144), and property (C4),

$$|U^g(s_1, s_2, v(s_1), v(s_2)) - U^f(s_1, s_2, v(s_1), v(s_2))| \le \delta_1/32. \tag{2.149}$$

It follows from (2.143), (2.147), and (2.148) that $U^f(s_1, s_2, v(s_1), v(s_2)) \le M_1$. By this inequality and (2.149), we have

$$U^g(s_1, s_2, v(s_1), v(s_2)) \le M_1 + \delta_1/32.$$

Combined with (2.146) this inequality implies that

$$I^g(s_1, s_2, v) \le U^g(s_1, s_2, v(s_1), v(s_2)) + \delta \le M_1 + \delta_1/32 + \delta. \tag{2.150}$$

In view of (2.150), (2.148), (2.144), and the property (C5), we have

$$|I^f(s_1, s_2, v) - I^g(s_1, s_2, v)| \le \delta_1/32.$$

By this inequality, (2.150), (2.149), and the choice of δ,

$$\begin{aligned} I^f(s_1, s_2, v) &\le I^g(s_1, s_2, v) + \delta_1/32 \le U^g(s_1, s_2, v(s_1), v(s_2)) + \delta + \delta_1/32 \\ &\le U^f(s_1, s_2, v(s_1), v(s_2)) + \delta_1/32 + \delta + \delta_1/32 \end{aligned}$$

and

$$I^f(s_1, s_2, v) \le U^f(s_1, s_2, v(s_1), v(s_2)) + 3\delta_1/32. \tag{2.151}$$

Hence, we have shown that the following property holds:

(C6) Inequality (2.151) is true for each pair of real numbers s_1, s_2 satisfying (2.148).

Let

$$\tau \in [T_1 + L_1 + 1, T_2 - L_1 - 1]. \tag{2.152}$$

By (2.152) and (2.145),

$$\tau - 1 - L_1,\ \tau + 1 + L_1 \in [T_1, T_2].$$

The property (C6) implies that

$$I^f(\tau-1-L_1, \tau-1, v) \le U^f(\tau-1-L_1, \tau-1, v(\tau-1-L_1), v(\tau-1)) + 3\delta_1/32, \tag{2.153}$$

$$I^f(\tau+1, \tau+1+L_1, v) \le U^f(\tau+1, \tau+1+L_1, v(\tau+1), v(\tau+1+L_1)) + 3\delta_1/32. \tag{2.154}$$

By (2.153), (2.154), (2.147), and the property (C3), there exists a pair of real numbers

$$t_1 \in [\tau - 1 - L_1, \tau - 1],\ t_2 \in [\tau + 1, \tau + 1 + L_1] \tag{2.155}$$

which satisfies

$$|X_f(t_i) - v(t_i)| \le \delta_0,\ i = 1, 2. \tag{2.156}$$

In view of property (C6),

$$I^f(\tau-1-L_1,\tau+1+L_1,v)$$
$$\leq U^f(\tau-1-L_1,\tau+1+L_1,v(\tau-1-L_1),v(\tau+1+L_1))+3\delta_1/32.$$

It follows from this inequality and (2.155) that

$$I^f(t_1,t_2,v)\leq U^f(t_1,t_2,v(t_1),v(t_2))+3\delta_1/32. \tag{2.157}$$

By (2.155), (2.156), (2.157), and property (C1),

$$|v(t)-X_f(t)|\leq\epsilon \text{ for all } t\in[t_1,t_2]$$

and

$$|v(\tau)-X_f(\tau)|\leq\epsilon. \tag{2.158}$$

Thus we have shown that the following property holds:

(C7) Inequality (2.158) is valid for each real number $\tau\in[T_1+L_1+1,T_2-L_1-1]$.

(Note that $[T_1+L,T_2-L]\subset[T_1+L_1+1,T_2-L_1-1]$.)

Let

$$|v(T_1)-X_f(T_1)|\leq\delta \tag{2.159}$$

and $\tau=T_1+L_1+1$. We have shown that there exists a real number

$$t_2\in[T_1+L_1+1,T_1+L_1+1+1+L_1] \tag{2.160}$$

for which

$$|X_f(t_2)-v(t_2)|\leq\delta_0 \tag{2.161}$$

(see (2.155), (2.156)).

In view of the property (C6), we have

$$I^f(T_1,T_1+2L_1+2,v)\leq U^f(T_1,T_1+2L_1+2,v(T_1),v(T_1+2L_1+2))+3\delta_1/2.$$

It follows from this inequality and (2.160) that

$$I^f(T_1,t_2,v)\leq U^f(T_1,t_2,v(T_1),v(t_2))+3\delta_1/2. \tag{2.162}$$

By (2.162), (2.161), (2.159), (2.160), and property (C1),

$$|v(t)-X_f(t)|\leq\epsilon,\ t\in[T_1,t_2].$$

(Note that $[T_1,T_1+L_1+1]\subset[T_1,t_2]$.) When combined with the property (C7), this implies that inequality (2.158) holds for each real number τ belonging to the interval $[T_1,T_2-L_1-1]$ which contains $[T_1,T_2-L]$.

Let

$$|v(T_2) - X_f(T_2)| \leq \delta \tag{2.163}$$

and $\tau = T_2 - L_1 - 1$. We have shown (see (2.155), (2.156)) that there exists a real number

$$t_1 \in [T_2 - L_1 - 2 - L_1, T_2 - L_1 - 2] \tag{2.164}$$

which satisfies

$$|X_v(t_1) - v(t_1)| \leq \delta_0. \tag{2.165}$$

It follows from (2.164) and property (C6) that

$$I^f(t_1, T_2, v) \leq U^f(t_1, T_2, v(t_1), v(T_2)) + 3\delta_1/2. \tag{2.166}$$

By (2.164), (2.163), (2.165), (2.166), and property (C1),

$$|v(t) - X_f(t)| \leq \epsilon$$

for any t in the interval $[t_1, T_2]$ which contains $[T_2 - L_1 - 2, T_2]$. When combined with the property (C7) this implies that inequality (2.158) holds for each real number τ belonging to the interval $[T_1 + L_1 + 1, T_2]$ which contains $[T_1 + L, T_2]$. This completes the proof of the theorem.

3

Autonomous Problems

In this chapter we study turnpike properties for autonomous variational problems with continuous integrands which belong to certain spaces of functions. In particular, we establish a turnpike property which means that restrictions of approximate solutions (on sufficiently large intervals), to any subinterval of a certain fixed length, are approximations of translations of the turnpike.

3.1 Problems with Continuous Integrands

We study the structure of approximate solutions of the variational problems:

$$\int_{T_1}^{T_2} f(z(t), z'(t))dt \to \min,\ z(T_1) = x,\ z(T_2) = y; \qquad (P)$$

$$z : [T_1, T_2] \to R^n \text{ is an absolutely continuous function,}$$

where $T_1 \geq 0$, $T_2 > T_1$, $x, y \in R^n$, and $f : R^n \times R^n \to R^1$ belongs to a space of autonomous integrands described below.

Denote by $|\cdot|$ the Euclidean norm in R^n. Let a be a positive constant and let $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that $\psi(t) \to +\infty$ as $t \to \infty$. We use a and ψ in order to define the space of integrands. Denote by $\mathcal{A}$ the set of all continuous functions $f : R^n \times R^n \to R^1$ which satisfy the following assumptions:

A(i) The function $f(x, \cdot) : R^n \to R^1$ is convex for each point $x \in R^n$.

A(ii) $f(x, u) \geq \max\{\psi(|x|), \psi(|u|)|u|\} - a$ for each $(x, u) \in R^n \times R^n$.

A(iii) For each pair of positive numbers M, ϵ, there exists a pair of positive numbers Γ, δ such that

$$|f(x_1, u_1) - f(x_2, u_2)| \leq \epsilon \max\{f(x_1, u_1), f(x_2, u_2)\}$$

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for each $u_1, u_2, x_1, x_2 \in R^n$ satisfying

$$|x_i| \le M,\ |u_i| \ge \Gamma\ (i = 1, 2), \max\{|x_1 - x_2|, |u_1 - u_2|\} \le \delta.$$

It is an elementary exercise to verify that a function $f = f(x, u) \in C^1(R^{2n})$ belongs to the set $\mathcal{A}$ if the function f satisfies assumptions A(i), A(ii) and there is an increasing function $\psi_0 : [0, \infty) \to [0, \infty)$ such that for each pair of points $x, u \in R^n$, we have

$$\max\{|\partial f/\partial x(x, u)|,\ |\partial f/\partial u(x, u)|\} \le \psi_0(|x|)(1 + \psi(|u|)|u|).$$

Note that the set $\mathcal{A}$ is a subset of the space of integrands $\mathcal{M}$ introduced in Sect. 2.1.

We equip the set $\mathcal{A}$ with the uniformity which is induced by the uniformity of the space $\mathcal{M}$ and determined by the base

$$\begin{aligned} E(N, \epsilon, \lambda) = \{(f, g) \in \mathcal{A} \times \mathcal{A} :\ & |f(x, u) - g(x, u)| \le \epsilon \\ & \text{for each } x, u \in R^n \text{ such that } |x|, |u| \le N \\ & \text{and } (|f(x, u)| + 1)(|g(x, u)| + 1)^{-1} \in [\lambda^{-1}, \lambda] \\ & \text{for each } x, u \in R^n \text{ such that } |x| \le N\}, \end{aligned}$$

where $N > 0$, $\epsilon > 0$, $\lambda > 1$.

It is clear that the uniform space $\mathcal{A}$ is Hausdorff and possesses a countable base. Thus the space $\mathcal{A}$ is metrizable (by a metric ρ). In Chap. 1 of [51] we show that the uniform space $\mathcal{A}$ is complete. We endow the space $\mathcal{A}$ with a topology induced by the metric ρ.

In this section which is based on [48], for any $f \in \mathcal{A}$, we define a nonempty compact set $\mathcal{D}(f) \subset R^n$ and show (see Theorems 3.2 and 3.3) that approximate solutions of problem (P) spend most of time in a small neighborhood of the set $\mathcal{D}(f)$.

We consider integral functionals

$$I^f(T_1, T_2, x) = \int_{T_1}^{T_2} f(x(t), x'(t))dt \tag{3.1}$$

where $f \in \mathcal{A}$, $-\infty < T_1 < T_2 < \infty$, and $x : [T_1, T_2] \to R^n$ is an absolutely continuous (a.c.) function.

For each integrand $f \in \mathcal{A}$, each pair of points $y, z \in R^n$, and each pair of real numbers $T_1, T_2 \in R^1$ satisfying $T_1 < T_2$, we put

$$\begin{aligned} U^f(T_1, T_2, y, z) = \inf\{I^f(T_1, T_2, x) :\ & x : [T_1, T_2] \to R^n \\ & \text{is an a.c. function such that } x(T_1) = y,\ x(T_2) = z\}. \end{aligned} \tag{3.2}$$

Clearly, the value $U^f(T_1, T_2, y, z)$ is finite for each $f \in \mathcal{A}$, each $y, z \in R^n$, and each $T_1 \in R^1$, $T_2 > T_1$.

Let $f \in \mathcal{A}$. We say that a locally absolutely continuous (a.c.) function $v : R^1 \to R^n$ is (f)-minimal if

$$\sup\{|v(t)| : t \in R^1\} < \infty$$

and if

$$I^f(T_1, T_2, v) = U^f(T_1, T_2, v(T_1), v(T_2))$$

for each pair of real numbers $T_1 \in R^1$, $T_2 > T_1$.

For each integrand $f \in \mathcal{A}$, denote by $\mathcal{M}(f)$ the set of all (f)-minimal functions $v : R^1 \to R^n$ and define

$$\mathcal{D}(f) = \cup\{v(R^1) : v \in \mathcal{M}(f)\}. \tag{3.3}$$

For each point $x \in R^n$ and each nonempty set $A \subset R^n$, put

$$d(x, A) = \inf\{|x - y| : y \in A\}. \tag{3.4}$$

In this chapter we prove the following results which were obtained in the work [48].

Theorem 3.1. *Let $f \in \mathcal{A}$. Then $\mathcal{M}(f) \neq \emptyset$ and $\mathcal{D}(f)$ is a bounded closed subset of R^n.*

Theorem 3.2. *Let $f \in \mathcal{A}$, $M, \epsilon > 0$. Then there exist a pair of positive numbers δ, L and a neighborhood $\mathcal{U}$ of the integrand f in the space $\mathcal{A}$ such that for each $g \in \mathcal{U}$, each real number $T \geq 2L$, and each a.c. function $v : [0, T] \to R^n$ which satisfies*

$$|v(0)|, |v(T)| \leq M, \ I^g(0, T, v) \leq U^g(0, T, v(0), v(T)) + \delta,$$

the inequality $d(v(t), \mathcal{D}(f)) \leq \epsilon$ is true for all $t \in [L, T - L]$.

Theorem 3.3. *Let $f \in \mathcal{A}$ and let M_0, M_1, ϵ be positive numbers. Then there exist an integer $q \geq 1$, a real number $L > 0$, and a neighborhood $\mathcal{U}$ of the integrand f in the space $\mathcal{A}$ such that the following assertion holds.*

For each integrand $g \in \mathcal{U}$, each real number $T \geq L$, and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$|v(0)|, |v(T)| \leq M_0, \ I^g(0, T, v) \leq U^g(0, T, v(0), v(T)) + M_1,$$

there exists a finite number of intervals $[a_i, b_i]$, $i = 1, \dots, p$ such that

$$p \leq q, \ a_{i+1} \geq b_i \text{ for all integers } i \text{ satisfying } 1 \leq i < p,$$

$$0 \leq b_i - a_i \leq L, \ i = 1, \dots, p$$

and

$$d(v(t), \mathcal{D}(f)) \leq \epsilon \text{ for all } t \in [0, T] \setminus \cup_{i=1}^{p} [a_i, b_i].$$

3.2 Preliminaries and Auxiliary Results

In this chapter we denote by $\mathrm{mes}(E)$ the Lebesgue measure of a Lebesgue-measurable set $E \subset R^q$ and by $\mathrm{Card}(A)$ the cardinality of a set A.

Let $f \in \mathcal{A}$ be given. Let us remind the following definition given in Sect. 2.1.

We say that a locally absolutely continuous (a.c.) function $x : [0,\infty) \to R^n$ is an (f)-*good function* if for any a.c. function $y : [0,\infty) \to R^n$, there exists a real number M_y depending on y such that

$$I^f(0,T,y) \geq M_y + I^f(0,T,x) \text{ for all } T \in (0,\infty).$$

Let $f \in \mathcal{A}$ be given. For any a.c. function $x : [0,\infty) \to R^n$, put

$$J(x) = \liminf_{T\to\infty} T^{-1} I^f(0,T,x). \tag{3.5}$$

Of special interest is the minimal long-run average cost growth rate

$$\mu(f) = \inf\{J(x) : \; x : [0,\infty) \to R^n \text{ is an a.c. function}\}. \tag{3.6}$$

Evidently, the value $\mu(f)$ is finite, and for every (f)-good function $x : [0,\infty) \to R^n$, we have

$$\mu(f) = J(x). \tag{3.7}$$

In [41] (see also Chap. 3 of [51]) we establish the following result.

Proposition 3.4. *For any a.c. function $x : [0,\infty) \to R^n$, either*

$$I^f(0,T,x) - T\mu(f) \to +\infty \text{ as } T \to \infty$$

or

$$\sup\{|I^f(0,T,x) - T\mu(f)| : \; T \in (0,\infty)\} < \infty. \tag{3.8}$$

Moreover (3.8) is valid if and only if the function x is (f)-good.

In our study we need the following useful results.

Proposition 3.5. *(Proposition 2.5 of [42], Chap. 1 of [51]). Assume that $f \in \mathcal{A}$, $M_1 > 0$, $-\infty < T_1 < T_2 < \infty$, $x_i : [T_1,T_2] \to R^n$, $i = 1,2,\ldots$ is a sequence of a.c. functions such that*

$$I^f(T_1,T_2,x_i) \leq M_1, \; i = 1,2,\ldots.$$

Then there exist a subsequence $\{x_{i_k}\}_{k=1}^\infty$ and an a.c. function $x : [T_1,T_2] \to R^n$ such that

$$I^f(T_1,T_2,x) \leq M_1, \; x_{i_k}(t) \to x(t) \text{ as } k \to \infty \text{ uniformly in } [T_1,T_2] \text{ and}$$

$$x'_{i_k} \to x' \text{ as } k \to \infty \text{ weakly in } L^1(R^n;(T_1,T_2)).$$

Proposition 3.6. *(Proposition 2.3 of [42], Chap. 1 of [51]). Let $M_1 > 0$, $0 < \tau_0 < \tau_1$. Then there exists a number $M_2 > 0$ such that for each $f \in \mathcal{A}$, each pair of numbers T_1, T_2 satisfying*

$$T_2 - T_1 \in [\tau_0, \tau_1]$$

and each a.c. function $x : [T_1, T_2] \to R^n$ which satisfies

$$I^f(T_1, T_2, x) \leq M_1,$$

the following relation holds:

$$|x(t)| \leq M_2, \; t \in [T_1, T_2].$$

Proposition 3.7. *(Proposition 5.2 of [41], Chap. 3 of [51]). Let $f \in \mathcal{A}$, $S_0, S_1 > 0$ and let $x : [0, \infty) \to R^n$ be an a.c. function such that $|x(t)| \leq S_0$ for all $t \in [0, \infty)$ and*

$$I^f(0, T, x) \leq U^f(0, T, x(0), x(T)) + S_1 \text{ for any } T > 0.$$

Then x is an (f)-good function.

Proposition 3.8. *(Theorem 6.1 of [41], Chap. 3 of [51]). Assume that $f \in \mathcal{A}$. Then the mapping $(T_1, T_2, x, y) \to U^f(T_1, T_2, x, y)$ is continuous for $T_1 \in R^1$, $T_2 \in (T_1, \infty)$, $x, y \in R^n$.*

3.3 Proof of Theorem 3.1

First we show that $\mathcal{M}(f) \neq \emptyset$. Proposition 2.16 implies that for each integer $k \geq 1$, there exists an a.c. function $x_k : [-k, k] \to R^n$ which satisfies

$$x_k(-k), x_k(k) = 0 \text{ and } I^f(-k, k, x_k) = U^f(-k, k, 0, 0). \tag{3.9}$$

It follows from (3.9) and Proposition 2.7 that there exists a positive number S_0 for which

$$|x_k(t)| \leq S_0 \text{ for all } t \in [-k, k], \; k = 1, 2, \ldots. \tag{3.10}$$

By (3.9), (3.10), and Proposition 2.9, for each integer $k \geq 1$, the sequence $\{I^f(-k, k, x_i)\}_{i=k}^{\infty}$ is bounded. Together with Proposition 3.5 this implies that there are a subsequence $\{x_{i_q}\}_{q=1}^{\infty}$ of $\{x_i\}_{i=1}^{\infty}$ and an a.c. function $x : R^1 \to R^n$ such that for each integer $k \geq 1$, we have

$$x_{i_q} \to x \text{ as } q \to \infty \text{ uniformly in } [-k, k], \tag{3.11}$$

$$x'_{i_q} \to x' \text{ as } q \to \infty \text{ weakly in } L^1(R^n; (-k, k))$$

and

$$I^f(-k,k,x) \le \liminf_{q\to\infty} I^f(-k,k,x_{i_q}). \tag{3.12}$$

It follows from (3.10) and (3.11) that

$$|x(t)| \le S_0 \text{ for all } t \in R^1. \tag{3.13}$$

In view of (3.12), (3.9), (3.11), and Proposition 3.8, for each integer $k \ge 1$,

$$I^f(-k,k,x) \le \liminf_{q\to\infty} U^f(-k,k,x_{i_q}(-k),x_{i_q}(k)) = U^f(-k,k,x(-k),x(k)).$$

By the inequality above,

$$I^f(T_1,T_2,x) = U^f(T_1,T_2,x(T_1),x(T_2)) \text{ for each } T_1 \in R^1 \text{ and each } T_2 > T_1.$$

Thus $x \in \mathcal{M}(f)$ by definition and $\mathcal{M}(f) \ne \emptyset$.

We claim that the set $\mathcal{D}(f)$ is bounded. Proposition 2.6 implies that there is a positive number S_1 satisfying

$$\limsup_{t\to\infty} |x(t)| < S_1 \tag{3.14}$$

for each (f)-good function $x : [0,\infty) \to R^n$. By Proposition 2.9, there exists a real number $S_2 > S_1 + 4$ such that

$$\sup\{|U^f(0,1,y,z)| : \ y,z \in R^n \text{ and } |y|,|z| \le S_1\} < S_2 - 4. \tag{3.15}$$

In view of Proposition 3.6, there exists a real number $S_3 > S_2$ such that for each a.c. function $x : [0,1] \to R^n$ satisfying $I^f(0,1,x) \le S_2$, we have

$$|x(t)| \le S_3 \text{ for all } t \in [0,1]. \tag{3.16}$$

Proposition 2.7 implies that there is a real number $S_* > S_3$ such that for each number $T \ge 1$ and each a.c. function $v : [0,T] \to R^n$ which satisfies

$$|v(0)|,|v(T)| \le S_3, \ I^f(0,T,v) \le U^f(0,T,v(0),v(T)) + 1, \tag{3.17}$$

we have

$$|v(t)| \le S_* \text{ for all } t \in [0,T]. \tag{3.18}$$

Let $x \in \mathcal{M}(f)$ be given. We claim that $|x(t)| \le S_*$ for all $t \in R^1$. For each nonnegative number t put $y_0(t) = x(t)$. In view of the inclusion $x \in \mathcal{M}(f)$, the definition of the set $\mathcal{M}(f)$ and Proposition 3.7, y_0 is an (f)-good function. Together with the choice of S_1 this implies that inequality (3.14) is valid.

We claim that

$$\liminf_{t\to-\infty} |x(t)| \le S_3. \tag{3.19}$$

Assume the contrary. Then

$$\liminf_{t \to -\infty} |x(t)| > S_3,$$

and there exists a negative number τ such that

$$|x(t)| > S_3 \text{ for all } t < \tau. \tag{3.20}$$

Since $x \in \mathcal{M}(f)$ there exists a positive number M_0 such that

$$|x(t)| \le M_0 \text{ for all } t \in R^1. \tag{3.21}$$

In view of inequality (3.14) there exists a number $\tau_0 > 1$ such that

$$|x(t)| < S_1 \text{ for all } t \ge \tau_0. \tag{3.22}$$

By Proposition 2.9, there exists a positive number S_4 such that

$$\sup\{|U^f(0, 1, y, z)| : \ y, z \in R^n \text{ and } |y|, |z| \le M_0\} + 2 \le S_4. \tag{3.23}$$

Fix an integer $L \ge 1$ such that

$$L > 4(2 + S_4). \tag{3.24}$$

Consider an a.c. function $y : [\tau - L, \tau] \to R^n$ such that

$$y(\tau - L) = x(\tau - L), \ y(\tau - L + 1) = x(\tau_0 + 1), \tag{3.25}$$

$$I^f(\tau - L, \tau - L + 1, y) \le U^f(\tau - L, \tau - L + 1, x(\tau - L), x(\tau_0 + 1)) + 1,$$

$$y(\tau - L + 1 + t) = x(\tau_0 + 1 + t), \ t \in [0, L - 2],$$

$$y(\tau) = x(\tau), \ I^f(\tau - 1, \tau, y) \le U^f(\tau - 1, \tau, x(\tau_0 + L - 1), x(\tau)) + 1.$$

Clearly,

$$y(\tau - L) = x(\tau - L), \ y(\tau) = x(\tau).$$

It is easy to see that

$$I^f(\tau - L, \tau, x) - I^f(\tau - L, \tau, y)$$

$$= \sum_{i=0}^{L-1} [I^f(\tau - L + i, \tau - L + i + 1, x) - I^f(\tau - L + i, \tau - L + i + 1, y)]. \tag{3.26}$$

By (3.20) and the choice of S_3 (see (3.16)), for each integer $i = 0, \dots, L - 1$, we have

$$I^f(\tau - L + i, \tau - L + i + 1, x) > S_2. \tag{3.27}$$

By (3.25) and (3.22),

$$|y(t)| \le S_1 \text{ for all } t \in [\tau - L + 1, \tau - 1]. \tag{3.28}$$

It follows from (3.28), (3.15), (3.25), (3.22), and the inclusion $x \in \mathcal{M}(f)$ that for each $i = 1, \dots, L-2$, we have

$$I^f(\tau - L + i, \tau - L + i + 1, y)$$
$$= U^f(\tau - L + i, \tau - L + i + 1, y(\tau - L + i), y(\tau - L + i + 1)) < S_2 - 4. \tag{3.29}$$

Relations (3.25) and (3.21) imply that

$$|y(\tau - L)|, |y(\tau - L + 1)|, |y(\tau - 1)|, |y(\tau)| \le M_0.$$

It follows from this inequality, (3.23), and (3.25) that

$$I^f(\tau - L, \tau - L + 1, y),\ I^f(\tau - 1, \tau, y) \le 1 + S_4. \tag{3.30}$$

By (3.26), (3.27), (3.30), (3.29), and (3.24),

$$I^f(\tau - L, \tau, x) - I^f(\tau - L, \tau, y)$$
$$= \sum_{i=0}^{L-1} I^f(\tau - L + i, \tau - L + i + 1, x) - \sum_{i=0}^{L-1} I^f(\tau - L + i, \tau - L + i + 1, y)$$
$$> S_2 L - 2(1 + S_4) - (S_2 - 4)(L - 2) = 4L - 2(1 + S_4) > 0.$$

This contradicts the inclusion $x \in \mathcal{M}(f)$. The contradiction we have reached proves inequality (3.19).

Let $h \in R^1$ be given. It follows from (3.19) and (3.14) that there exists a pair of real numbers $T_1 < h - 2$, $T_2 > h + 2$ such that

$$|v(T_i)| \le S_3,\ i = 1, 2.$$

It follows from this inequality, the choice of S_* (see (3.17), (3.18)), and the inclusion $x \in \mathcal{M}(f)$ that

$$|x(h)| \le S_*.$$

Thus $|x(h)| \le S_*$ for all $h \in R^1$. We have shown that

$$|z| \le S_* \text{ for all } z \in \mathcal{D}(f). \tag{3.31}$$

We claim that $\mathcal{D}(f)$ is closed. Assume that

$$\{z_i\}_{i=1}^{\infty} \subset \mathcal{D}(f) \text{ and } z = \lim_{i \to \infty} z_i. \tag{3.32}$$

For each integer $i \geq 1$ there exists $v_i \in \mathcal{M}(f)$ which satisfies that $z_i \in v_i(R^1)$. We may assume without any loss of generality that

$$z_i = v_i(0), \ i = 1, 2, \dots. \tag{3.33}$$

Let $k \geq 1$ be an integer. It follows from (3.31) and Proposition 2.9 that the sequence $\{I^f(-k, k, v_i)\}_{i=1}^{\infty}$ is bounded. Together with Proposition 3.5 this implies that there exist a subsequence $\{v_{i_q}\}_{q=1}^{\infty}$ of $\{v_i\}_{i=1}^{\infty}$ and an a.c. function $v : R^1 \to R^n$ such that for each integer $k \geq 1$, we have

$$\begin{aligned} &v_{i_q} \to v \text{ as } q \to \infty \text{ uniformly in } [-k, k], \\ &v'_{i_q} \to v' \text{ as } q \to \infty \text{ weakly in } L^1(R^n; (-k, k)), \\ &I^f(-k, k, v) \leq \liminf_{q \to \infty} I^f(-k, k, v_{i_q}). \end{aligned} \tag{3.34}$$

By (3.34), Proposition 3.8, and the inclusion $v_i \in \mathcal{M}(f)$, $i = 1, 2, \dots$, for each integer $k \geq 1$, we have

$$\begin{aligned} I^f(-k, k, v) &\leq \liminf_{q \to \infty} U^f(-k, k, v_{i_q}(-k), v_{i_q}(k)) \\ &= U^f(-k, k, v(-k), v(k)). \end{aligned}$$

Thus for each integer $k \geq 1$, we have

$$I^f(-k, k, v) = U^f(-k, k, v(-k), v(k))$$

and $v \in \mathcal{M}(f)$. It follows from (3.34), (3.33), and (3.32) that

$$v(0) = \lim_{q \to \infty} v_{i_q}(0) = \lim_{q \to \infty} z_{i_q} = z$$

and that $z \in \mathcal{D}(f)$. Therefore $\mathcal{D}(f)$ is closed, as claimed. This completes the proof of Theorem 3.1.

3.4 Proof of Theorem 3.2

Assume that the theorem is not true. Then for each integer $i \geq 1$ there exist an integrand $f_i \in \mathcal{A}$ such that

$$\rho(f, f_i) \leq 1/i, \tag{3.35}$$

a real number

$$T_i \geq 2i, \tag{3.36}$$

an a.c. function $v_i : [0, T_i] \to R^n$ which satisfies

$$|v_i(0)|, |v_i(T_i)| \le M,\ I^{f_i}(0, T_i, v_i) \le U^{f_i}(0, T_i, v_i(0), v_i(T_i)) + 1/i, \tag{3.37}$$

and a real number

$$t_i \in [i, T_i - i] \tag{3.38}$$

such that

$$d(v_i(t_i), \mathcal{D}(f)) > \epsilon. \tag{3.39}$$

For each integer $i \ge 1$ put

$$u_i(t) = v_i(t + t_i),\ t \in [-t_i, T_i - t_i]. \tag{3.40}$$

It follows from (3.38)–(3.40) that

$$[-i, i] \subset [-t_i, T_i - t_i] \text{ for all natural numbers } i \tag{3.41}$$

and

$$d(u_i(0), \mathcal{D}(f)) > \epsilon \text{ for all natural numbers } i. \tag{3.42}$$

By (3.40) and (3.37) that for all integers $i \ge 1$, we have

$$|u_i(-t_i)|,\ |u_i(T_i - t_i)| \le M \tag{3.43}$$

and

$$I^{f_i}(-t_i, T_i - t_i, u_i) \le U^{f_i}(-t_i, T_i - t_i, u_i(t_i), u_i(T_i - t_i)) + 1/i. \tag{3.44}$$

Proposition 2.7, (3.43), (3.36), and (3.35) imply that there exists a positive number S such that

$$|u_i(t)| \le S,\ t \in [-t_i, T_i - t_i],\ i = 1, 2, \ldots. \tag{3.45}$$

Let $k \ge 1$ be an integer. By (3.41), (3.45), and Proposition 2.9,

$$\sup\{|U^f(-k, k, u_i(-k), u_i(k))| :\ i \ge k \text{ is an integer } \} < \infty. \tag{3.46}$$

It follows from (3.45), (3.35), and Proposition 2.8 that

$$\lim_{i\to\infty} |U^{f_i}(-k, k, u_i(-k), u_i(k)) - U^f(-k, k, u_i(-k), u_i(k))| = 0. \tag{3.47}$$

By (3.44),

$$\lim_{i\to\infty} [I^{f_i}(-k, k, u_i) - U^{f_i}(-k, k, u_i(-k), u_i(k))] = 0. \tag{3.48}$$

It follows from (3.48), (3.47), and (3.46) that the sequence $\{I^{f_i}(-k,k,u_i)\}_{i=k}^{\infty}$ is bounded. Together with inequality (3.35) and Proposition 2.10 this implies that

$$\lim_{i\to\infty} |I^{f_i}(-k,k,u_i) - I^f(-k,k,u_i)| = 0. \tag{3.49}$$

Thus the sequence $\{I^f(-k,k,u_i)\}_{i=k}^{\infty}$ is bounded for any integer $k \geq 1$. Together with Proposition 3.5, this implies that there exist a subsequence $\{u_{i_q}\}_{q=1}^{\infty}$ of $\{u_i\}_{i=1}^{\infty}$ and an a.c. function $u : R^1 \to R^n$ such that for each integer $k \geq 1$, we have

$$u_{i_q} \to u \text{ as } q \to \infty \text{ uniformly in } [-k,k], \tag{3.50}$$

$$u'_{i_q} \to u' \text{ as } q \to \infty \text{ weakly in } L^1(R^n; (-k,k)),$$

$$I^f(-k,k,u) \leq \liminf_{q\to\infty} I^f(-k,k,u_{i_q}). \tag{3.51}$$

It follows from (3.50) and (3.45) that

$$|u(t)| \leq S \text{ for all } t \in R^1. \tag{3.52}$$

In view of (3.51), (3.49), (3.48), (3.47), and Proposition 3.8, for each integer $k \geq 1$, we have

$$I^f(-k,k,u) \leq \liminf_{q\to\infty} I^f(-k,k,u_{i_q}) \leq \liminf_{q\to\infty} I^{f_{i_q}}(-k,k,u_{i_q})$$

$$= \liminf_{q\to\infty} [U^{f_{i_q}}(-k,k,u_{i_q}(-k),u_{i_q}(k))]$$

$$= \liminf_{q\to\infty} [U^f(-k,k,u_{i_q}(-k),u_{i_q}(k))] = U^f(-k,k,u(-k),u(k)).$$

Thus we have shown that $u \in \mathcal{M}(f)$. Relations (3.50) and (3.42) imply that $d(u(0), \mathcal{D}(f)) \geq \epsilon$, a contradiction. The contradiction we have reached completes the proof of the theorem.

3.5 Proof of Theorem 3.3

Proposition 2.7 implies that there is a neighborhood $\mathcal{U}_0$ of the integrand f in the space $\mathcal{A}$ and a real number $M_2 > M_1 + M_0$ such that the following property holds:

(P0) For each integrand $g \in \mathcal{U}_0$, each real number $T \geq 1$ and each a.c. function $v : [0,T] \to R^n$ which satisfies

$$|v(0)|, |v(T)| \leq M_0, \; I^g(0,T,v) \leq U^g(0,T,v(0),v(T)) + M_3,$$

we have

$$|v(t)| \le M_2 \text{ for all } t \in [0, T].$$

Theorem 3.2 implies that there exist a neighborhood $\mathcal{U}$ of the integrand f in the space $\mathcal{A}$ and a pair of real numbers $L_0 > 1$ and $\delta_0 > 0$ such that $\mathcal{U} \subset \mathcal{U}_0$ and that the following property holds:

(P1) For each integrand $g \in \mathcal{U}$, each real number $T \ge 2L_0$, and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$|v(0)|, |v(T)| \le M_2, \ I^g(0, T, v) \le U^g(0, T, v(0), v(T)) + \delta_0, \tag{3.53}$$

we have

$$d(v(t), \mathcal{D}(f)) \le \epsilon \text{ for all } t \in [L_0, T - L_0]. \tag{3.54}$$

Put

$$L = 2L_0 \tag{3.55}$$

and fix an integer

$$q > 3(M_1\delta_0^{-1} + 3). \tag{3.56}$$

Let $g \in \mathcal{U}$, $T \ge L$ and let an a.c. function $v : [0, T] \to R^n$ satisfy

$$|v(0)|, |v(T)| \le M_0, \ I^g(0, T, v) \le U^g(0, T, v(0), v(T)) + M_1. \tag{3.57}$$

It follows from (3.57) and the property (P0) that

$$|v(t)| \le M_2 \text{ for all } t \in [0, T]. \tag{3.58}$$

It is easy to see that there exist a finite number of real numbers t_i, $i = 1, \dots, p$ such that $t_1 = 0$, $t_p = T$; if an integer i satisfies $1 \le i < p$, then

$$I^g(t_i, t_{i+1}, v) = U^g(t_i, t_{i+1}, v(t_i), v(t_{i+1})) + \delta_0 \tag{3.59}$$

and

$$I^g(t_{p-1}, t_p, v) \le U^g(t_{p-1}, t_p, v(t_{p-1}), v(t_p)) + \delta_0. \tag{3.60}$$

By (3.57) and (3.59),

$$M_1 \ge I^g(0, T, v) - U^g(0, T, v(0), v(T))$$

$$\ge \sum_{i=1}^{p-1} [I^g(t_i, t_{i+1}, v) - U^g(t_i, t_{i+1}, v(t_i), v(t_{i+1}))] \ge \delta_0(p-2)$$

and

$$p \le 2 + M_1/\delta_0. \tag{3.61}$$

Define

$$\mathcal{A} = \{[t_i, t_{i+1}] : \ i \in \{1, \dots, p-1\} \text{ and } t_{i+1} - t_i \le L\}$$
$$\cup \{[t_i, t_i + L_0] : \ i \in \{1, \dots, p-1\} \text{ and } t_{i+1} - t_i > L\}$$
$$\cup \{[t_{i+1} - L_0, t_{i+1}] : \ i \in \{1, \dots, p-1\} \text{ and } t_{i+1} - t_i > L\}. \tag{3.62}$$

It follows from (3.61) and (3.56) that

$$\mathrm{Card}(A) \le 3p \le 3(2 + M_1/\delta_0) < q. \tag{3.63}$$

In view of (3.62) and (3.55), for each $e \in A$, we have

$$\mathrm{mes}(e) \le L. \tag{3.64}$$

Assume that

$$t \in [0, T] \setminus \cup\{e : \ e \in A\}. \tag{3.65}$$

Then there exists an integer $j \in \{1, \dots, p-1\}$ such that

$$t_{j+1} - t_j > L, \ t \in [t_j + L_0, t_{j+1} - L_0]. \tag{3.66}$$

By (3.66), (3.58), (3.59), (3.60), (3.55), and the property (P1) applied to the restriction of v on $[t_j, t_{j+1}]$,

$$d(v(t), \mathcal{D}(f)) \le \epsilon.$$

This completes the proof of the theorem.

3.6 Problems with Smooth Integrands

We continue to study the structure of approximate solutions of the variational problems

$$\int_0^T f(z(t), z'(t))dt \to \min, \ z(0) = x, \ z(T) = y, \tag{P}$$

$$z : \ [0, T] \to R^n \text{ is an absolutely continuous function,}$$

where $T > 0$, $x, y \in R^n$, and $f : R^n \times R^n \to R^1$ belongs to a space of integrands described below.

Recall that we denote by $|\cdot|$ the Euclidean norm in R^n, a is a positive constant, and $\psi : [0, \infty) \to [0, \infty)$ is an increasing function such that $\psi(t) \to +\infty$ as $t \to \infty$. We consider the set $\mathcal{A}$ introduced in Sect. 3.1 which consists of all continuous functions $f : \ R^n \times R^n \to R^1$ satisfying the following assumptions:

A(i) For each $x \in R^n$, the function $f(x, \cdot) : R^n \to R^1$ is convex.

A(ii) $f(x, u) \ge \max\{\psi(|x|), \psi(|u|)|u|\} - a$ for each $(x, u) \in R^n \times R^n$.

A(iii) For each pair of positive numbers M, ϵ, there exists a pair of positive numbers Γ, δ such that

$$|f(x_1,u_1) - f(x_2,u_2)| \le \epsilon \max\{f(x_1,u_1), f(x_2,u_2)\}$$

for each $u_1, u_2, x_1, x_2 \in R^n$ which satisfy

$$|x_i| \le M,\ i = 1,2,\ |u_i| \ge \Gamma,\ i = 1,2, \quad |x_1 - x_2|,\ |u_1 - u_2| \le \delta.$$

We equip the set $\mathcal{A}$ with the uniformity which is defined in Sect. 3.1 and determined by the base

$$E(N,\epsilon,\lambda) = \{(f,g) \in \mathcal{A} \times \mathcal{A} :\ |f(x,u) - g(x,u)| \le \epsilon$$

$$\text{for all } u, x \in R^n \text{ such that } |x|, |u| \le N\}$$

$$\cap\{(f,g) \in \mathcal{A} \times \mathcal{A} :\ (|f(x,u)| + 1)(|g(x,u)| + 1)^{-1} \in [\lambda^{-1}, \lambda]$$

$$\text{for all } x, u \in R^n \text{ such that } |x| \le N\},$$

where $N, \epsilon > 0$ and $\lambda > 1$. Note that the uniform space $\mathcal{A}$ is metrizable and complete.

We study integral functionals

$$I^f(T_1, T_2, x) = \int_{T_1}^{T_2} f(x(t), x'(t))dt \tag{3.67}$$

where $f \in \mathcal{A}$, $-\infty < T_1 < T_2 < \infty$, and $x :\ [T_1, T_2] \to R^n$ is an absolutely continuous (a.c.) function.

Recall that for each integrand $f \in \mathcal{A}$, each pair of points $y, z \in R^n$, and each pair of real numbers T_1, T_2 satisfying $0 \le T_1 < T_2$,

$$U^f(T_1, T_2, y, z) = \inf\{I^f(T_1, T_2, x) :\ x :\ [T_1, T_2] \to R^n \tag{3.68}$$

$$\text{is an a.c. function satisfying } x(T_1) = y,\ x(T_2) = z\}.$$

Clearly, the value $U^f(T_1, T_2, y, z)$ is finite for each integrand $f \in \mathcal{A}$, each pair of points $y, z \in R^n$, and each pair of real numbers T_1, T_2 satisfying $0 \le T_1 < T_2$.

Let $f \in \mathcal{A}$ be given. For any a.c. function $x : [0, \infty) \to R^n$, we put

$$J(x) = \liminf_{T \to \infty} T^{-1} I^f(0, T, x) \tag{3.69}$$

and

$$\mu(f) = \inf\{J(x) :\ x : [0, \infty) \to R^n \text{ is an a.c. function}\}. \tag{3.70}$$

It is clear that the value $\mu(f)$ is finite. By a simple modification of the proof of Proposition 4.4 in [22] (see Theorems 8.1 and 8.2 of [41] and Chap. 3 of [51]), we obtained the representation formula

$$U^f(0,T,x,y) = T\mu(f) + \pi^f(x) - \pi^f(y) + \theta_T^f(x,y), \tag{3.71}$$

$$x, y \in R^n, \ T \in (0,\infty),$$

where $\pi^f : R^n \to R^1$ is a continuous function and $(T,x,y) \to \theta_T^f(x,y) \in R^1$ is a continuous nonnegative function defined for $T > 0$, $x, y \in R^n$,

$$\pi^f(x) = \inf\{\liminf_{T\to\infty}[I^f(0,T,v) - \mu(f)T] : \ v : [0,\infty) \to R^n \tag{3.72}$$

$$\text{is an a.c. function satisfying } v(0) = x\}, \ x \in R^n,$$

and for every positive number T, every point $x \in R^n$, there exists a point $y \in R^n$ such that $\theta_T^f(x,y) = 0$.

We use the notion of an (f)-good function introduced in Sect. 3.2. In Chap. 3 of [51] we show that for each integrand $f \in \mathcal{A}$ and each point $z \in R^n$ there exists an (f)-good function $v : [0,\infty) \to R^n$ such that $v(0) = z$.

We denote $d(x,B) = \inf\{|x-y| : \ y \in B\}$ for any point $x \in R^n$ and any nonempty set $B \subset R^n$ and denote by $\mathrm{dist}(A,B)$ the distance in the Hausdorff metric for any pair of two nonempty sets $A \subset R^n$ and $B \subset R^n$. For every bounded function $x : [0,\infty) \to R^n$ we set

$$\Omega(x) = \{y \in R^n : \ \text{there exists a sequence } \{t_i\}_{i=1}^{\infty} \subset (0,\infty) \tag{3.73}$$

$$\text{for which } t_i \to \infty, \ x(t_i) \to y \text{ as } i \to \infty\}.$$

We say that an integrand $f \in \mathcal{A}$ has an asymptotic turnpike property, or briefly ATP, if $\Omega(v_2) = \Omega(v_1)$ for all pairs of (f)-good functions $v_i : [0,\infty) \to R^n$, $i = 1,2$ [51].

In [41] and Chap. 3 of [51] we establish the existence of a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{A}$ such that each integrand $f \in \mathcal{F}$ possesses ATP.

It follows from Propositions 2.6 and 3.4 that for each integrand $f \in \mathcal{A}$ which possesses ATP, there exists a nonempty compact set $H(f) \subset R^n$ such that $\Omega(v) = H(f)$ for all (f)-good functions $v : [0,\infty) \to R^n$.

Denote by $\mathcal{N}$ the set of all functions $f \in C^2(R^{2n})$ which satisfy the following assumptions:

$$\partial f/\partial u_i \in C^2(R^{2n}) \text{ for } i = 1,\dots,n;$$

the matrix $(\partial^2 f/\partial u_i \partial u_j)(x,u)$, $i,j = 1,\dots,n$ is positive definite for all $(x,u) \in R^{2n}$;

$$f(x,u) \ge \max\{\psi(|x|), \ \psi(|u|)|u|\} - a \text{ for all } (x,u) \in R^n \times R^n;$$

there exist a real number $c_0 > 1$ and monotone increasing functions $\phi_i : [0,\infty) \to [0,\infty)$, $i = 0, 1, 2$ such that

$$\phi_0(t)/t \to \infty \text{ as } t \to \infty,$$

$$f(x,u) \geq \phi_0(c_0|u|) - \phi_1(|x|),\ x, u \in R^n,$$

$$\max\{|\partial f/\partial x_i(x,u)|,\ |\partial f/\partial u_i(x,u)|\}$$

$$\leq \phi_2(|x|)(1 + \phi_0(|u|)),\ x, u \in R^n,\ i = 1, \dots, n.$$

It is not difficult to see that $\mathcal{N} \subset \mathcal{A}$.

In [43] (see also Chap. 5 of [51]) we establish the following result which shows that if an integrand $f \in \mathcal{N}$ possesses ATP, then a turnpike property holds with the turnpike $H(f)$.

Theorem 3.9. *Assume that an integrand $f \in \mathcal{N}$ has ATP and that $\epsilon, K > 0$. Then there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ and numbers $M > K$, $l_0 > l > 0$, $\delta > 0$ such that for each $g \in \mathcal{U}$, each $T \geq 2l_0$, and each a.c. function $v : [0, T] \to R^n$ which satisfies*

$$|v(0)|, |v(T)| \leq K,\ I^g(0, T, v) \leq U^g(0, T, v(0), v(T)) + \delta,$$

the inequality $|v(t)| \leq M$ holds for all $t \in [0, T]$ and

$$dist(H(f), \{v(t) :\ t \in [\tau, \tau + l]\}) \leq \epsilon \tag{3.74}$$

for each $\tau \in [l_0, T - l_0]$. Moreover, if $d(v(0), H(f)) \leq \delta$, then (3.74) holds for each $\tau \in [0, T - l_0]$, and if $d(v(T), H(f)) \leq \delta$, then (3.74) holds for each $\tau \in [l_0, T - l]$.

For each integrand $f \in \mathcal{A}$, each pair of numbers $T_1, T_2 \in R^1$ such that $T_1 < T_2$, and each a.c. function $x : [T_1, T_2] \to R^n$, put

$$\sigma^f(T_1, T_2, x) = I^f(T_1, T_2, x) - (T_2 - T_1)\mu(f) - \pi^f(x(T_1)) + \pi^f(x(T_2)). \tag{3.75}$$

Since the function θ^f_T is nonnegative it follows from (3.71) and (3.75) that the functional $\sigma^f(\cdot,\cdot,\cdot)$ is also nonnegative.

Let $f \in \mathcal{A}$ be given. Following [27, 28], we say that an a.c. function $v : D \to R^n$ where D is either R^1 or $[0,\infty)$ is (f)-perfect if $\sigma^f(T_1, T_2, v) = 0$ for each pair of real numbers $T_1, T_2 \in D$ such that $T_1 < T_2$.

For any integrand $f \in \mathcal{A}$ which possesses ATP we denote by $\sigma(f)$ the set of all a.c. functions $v : R^1 \to H(f)$ such that

$$\sigma^f(T_1, T_2, v) = 0 \text{ for each } T_1 \in R^1 \text{ and each } T_2 > T_1. \tag{3.76}$$

In this chapter we will prove the following two results which were obtained in [50].

Proposition 3.10. *Let $f \in \mathcal{A}$ have ATP and $x \in H(f)$. Then there exists $v \in \sigma(f)$ such that $v(0) = x$.*

Proposition 3.11. *Let $f \in \mathcal{N}$, $-\infty < T_1 < T_2 < \infty$, and let $v_i : [T_1, T_2] \to R^n$, $i = 1, 2$ be a.c. functions which satisfy*

$$\sigma^f(T_1, T_2, v_i) = 0, \ i = 1, 2 \tag{3.77}$$

and $v_1(s) = v_2(s)$ with some $s \in (T_1, T_2)$. Then $v_1(s) = v_2(s)$ for all $t \in [T_1, T_2]$.

Corollary 3.12. *Let $f \in \mathcal{N}$ possess ATP. Then for each $x \in H(f)$ there exists a unique function $v \in \sigma(f)$ which satisfies $v(0) = x$.*

Let $f \in \mathcal{A}$ be given. We say that an a.c. function $v : R^1 \to R^n$ is c-optimal with respect to f [27, 28] if for each pair of real numbers $T_1 \in R^1$, $T_2 > T_1$, we have

$$I^f(T_1, T_2, v) = U^f(0, T_2 - T_1, v(T_1), v(T_2)).$$

The next proposition obtained in [50] will be proved in Sect. 3.8.

Proposition 3.13. *Let $f \in \mathcal{N}$ possess ATP. Then an a.c. function $v : R^1 \to H(f)$ belongs to $\sigma(f)$ if and only if it is c-optimal with respect to f.*

The following theorem which was established in [50] generalizes Theorem 3.9. It shows that if an integrand $f \in \mathcal{N}$ has ATP, then the strong version of the turnpike property holds and the turnpike is any $w \in \sigma(f)$.

Theorem 3.14. *Let an integrand $f \in \mathcal{N}$ possess ATP and let ϵ, K, l be positive numbers. Then there exist a neighborhood $\mathcal{U}$ of the integrand f in the space $\mathcal{A}$ and real numbers $\delta > 0$, $K_1 > K$, $l_2 \geq l_1 > l$ such that for each integrand $g \in \mathcal{U}$, each real number $T \geq 2l_2$, and each a.c. function $v : [0, T] \to R^n$ which satisfies*

$$|v(0)|, |v(T)| \leq K, \ I^g(0, T, v) \leq U^g(0, T, v(0), v(T)) + \delta,$$

the inequality $|v(t)| \leq K_1$ is true for all $t \in [0, T]$ and the following assertion holds:

There exists a pair of real numbers $\tau_1 \in [l, l_2]$, $\tau_2 \in [T - l_2, T - l]$ such that for each $\tau \in [\tau_1, \tau_2]$ and each $w \in \sigma(f)$, there exists a real number $s \in [0, l_1]$ such that

$$|v(\tau + t) - w(s + t)| \leq \epsilon \text{ for all } t \in [-l, l].$$

Moreover, if $d(v(0), H(f)) \leq \delta$, then $\tau_1 = l$, and if $d(v(T), H(f)) \leq \delta$, then $\tau_2 = T - l$.

Theorem 3.14 will be proved in Sect. 3.9.

Now we will define complete metric spaces of integrands and establish that most their elements (in the sense of Baire category) possess the strong version of the turnpike property.

Let $k \geq 3$ be an integer. Denote by $\mathcal{A}_k$ the set of all integrands $f \in \mathcal{A} \cap C^k(R^{2n})$. For any $p = (p_1, \ldots, p_{2n}) \in \{0, \ldots, k\}^{2n}$, put $|p| = \sum_{i=1}^{2n} p_i$. For each integrand $f \in C^k(R^{2n})$ and each $p = (p_1, \ldots, p_{2n}) \in \{0, \ldots, k\}^{2n}$ with $|p| \leq k$, define

$$D^p f = \partial^{|p|} f / \partial y_1^{p_1} \ldots \partial y_{2n}^{p_{2n}}.$$

We equip the set $\mathcal{A}_k$ with the uniformity determined by the base

$$E_k(\epsilon) = \{(f, g) \in \mathcal{A}_k \times \mathcal{A}_k : \ |D^p f(x, u) - D^p g(x, u)| \leq \epsilon$$

$$\text{for all } u, x \in R^n \text{ and all } p \in \{0, \ldots, k\}^{2n} \text{ satisfying } |p| \leq k\},$$

where ϵ is a positive number. (Here $D^0 f = f$.)

It is easy to see that the uniform spaces $\mathcal{A}_k$ is metrizable (by a metric ρ_k) and complete. Put $\mathcal{N}_k = \mathcal{N} \cap \mathcal{A}_k$ and denote by $\bar{\mathcal{N}}_k$ the closure of $\mathcal{N}_k$ in the space $(\mathcal{A}_k, \rho_k)$. It is clear that there exists a subset $\mathcal{F}_{k1} \subset \bar{\mathcal{N}}_k$ which is a countable intersection of open everywhere dense subsets of the space $(\bar{\mathcal{N}}_k, \rho_k)$ such that $\mathcal{F}_{k1} \subset \mathcal{N}_k$.

The next theorem, which was obtained in [50], will be proved in Sect. 3.10.

Theorem 3.15. *There exists a set $\mathcal{F}_{k0} \subset \bar{\mathcal{N}}_k$ which is a countable intersection of open everywhere dense subsets of $(\bar{\mathcal{N}}_k, \rho_k)$ such that each $f \in \mathcal{F}_{k0}$ possesses ATP.*

Theorem 3.15 and the inclusion $\mathcal{F}_{k1} \subset \mathcal{N}_k$ imply the following result.

Theorem 3.16. *There exists a set $\mathcal{F} \subset \mathcal{N}_k$ which is a countable intersection of open everywhere dense subsets of $(\bar{\mathcal{N}}_k, \rho_k)$ such that each $f \in \mathcal{F}$ has ATP.*

Theorems 3.14 and 3.16 imply that each integrand $f \in \mathcal{F}$ possesses the strong version of the turnpike property.

3.7 Auxiliary Results

In this section we collect several auxiliary results which will be used in this chapter.

Proposition 3.17. *(Proposition 5.1 of [41].) Let $g \in \mathcal{A}$, $y : [0, \infty) \to R^n$ be a (g)-good function and let $\epsilon > 0$. Then there exists $T_0 > 0$ such that for each $T \geq T_0$ and each $\bar{T} > T$,*

$$I^g(T, \bar{T}, y) \leq U^g(T, \bar{T}, y(T), y(\bar{T})) + \epsilon.$$

Proposition 3.18. *Assume that $g \in \mathcal{A}$, $y : [0, \infty) \to R^n$ is (g)-good function and that ϵ is a positive number. Then there exists a positive number $T(\epsilon)$ such that for each $T_1 \geq T(\epsilon)$ and each $T_2 > T_1$, the following inequality holds:*

$$\sigma^f(T_1, T_2, v) \leq \epsilon.$$

Proof. Propositions 2.6 and 3.4 imply that $\sup\{|y(t)| : \ t \in [0,\infty)\} < \infty$. Hence

$$\sup\{\sigma^f(0,T,y) : \ T \in (0,\infty)\} =$$
$$\sup\{I^f(0,T,y) - T\mu(f) - \pi^f(y(0)) + \pi^f(y(T)) : \ T \in (0,\infty)\} < \infty.$$

This equality implies the validity of Proposition 3.18.

Proposition 3.19. *(Proposition 7.1 of [44].) Let $f \in \mathcal{N}$, $x, y \in R^n$, $T_1 \in [0,\infty)$, $T_2 > T_1$, and let $w : [T_1,T_2] \to R^n$ be an a.c. function such that*

$$w(T_1) = x, \ w(T_2) = y, \ I^f(T_1,T_2,w) = U^f(T_1,T_2,x,y).$$

Then $w \in C^2([T_1,T_2]; R^n)$, and for each $i \in \{1,\dots,n\}$ and each $t \in [T_1,T_2]$,

$$\begin{aligned}(\partial f/\partial x_i)(w(t),w'(t)) &= (d/dt)(\partial f/\partial u_i)(w(t),w'(t))\\ &= \sum_{j=1}^{n}(\partial^2 f/\partial u_i \partial x_j)(w(t),w'(t))w_j'(t)\\ &+ \sum_{j=1}^{n}(\partial^2 f/\partial u_i \partial u_j)(w(t),w'(t))w_j''(t). \end{aligned} \tag{3.78}$$

Proposition 3.20. *(Corollary 2.1 of [42] and Chap. 1 of [51].) For each $f \in \mathcal{A}$, each pair of numbers T_1, T_2 satisfying $0 \le T_1 < T_2$, and each $z_1, z_2 \in R^n$, there is an a.c. function $x : [T_1,T_2] \to R^n$ such that $x(T_i) = z_i$, $i = 1,2$ and $I^f(T_1,T_2,x) = U^f(T_1,T_2,z_1,z_2)$.*

Proposition 3.21. *(Lemma 4.4 of [43] and Chap. 5 of [51].) Let $f \in \mathcal{N}$ have ATP and let $\epsilon > 0$. Then there exists a number $q \ge 8$ such that for each $h_1, h_2 \in H(f)$, there exists an a.c. function $v : [0,q] \to R^n$ which satisfies*

$$v(0) = h_1, \ v(q) = h_2, \ \sigma^f(0,q,v) \le \epsilon.$$

Proposition 3.22. *(Lemma 10.2 of [41] and Chap. 3 of [51].) Assume that $f \in \mathcal{A}$ possesses ATP, $\epsilon \in (0,1)$, and $K_0, M_0 > 0$. Then there are a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ and a number $S_0 > 0$ such that for each $g \in \mathcal{U}$, each $\tau \ge 0$, and each a.c. function $x : [\tau,\tau+S_0] \to R^n$ satisfying*

$$|x(\tau)|, \ |x(\tau+S_0)| \le K_0,$$

$$I^g(\tau,\tau+S_0,x) \le U^g(\tau,\tau+S_0,x(\tau),x(\tau+S_0)) + M_0,$$

there is $t \in [\tau,\tau+S_0]$ for which $d(x(t),H(f)) \le \epsilon$.

3.8 Proofs of Propositions 3.10, 3.11, and 3.13

Proof of Proposition 3.10: Let an (f)-good function $u : [0, \infty) \to R^n$ be given. Since f possesses ATP, we have $\Omega(u) = H(f)$. Propositions 2.6 and 3.4 imply that

$$\sup\{|u(t)| : t \in [0, \infty)\} < \infty. \tag{3.79}$$

There exists a sequence of positive numbers $t_i \to \infty$ as $i \to \infty$ satisfying

$$u(t_i) \to x \text{ as } i \to \infty. \tag{3.80}$$

In view of Proposition 3.17 the following property holds:

(a) For each $\epsilon > 0$ there is a positive number $T(\epsilon)$ such that for each real number $T_1 \geq T(\epsilon)$ and each real number $T_2 > T_1$, we have

$$I^f(T_1, T_2, u) \leq U^f(T_1, T_2, u(T_1), u(T_2)) + \epsilon.$$

Proposition 3.18 implies that the following property holds:

(b) For each positive number ϵ there exists a positive number $T(\epsilon)$ such that for each real number $T_1 \geq T(\epsilon)$ and each real number $T_2 > T_1$, we have

$$\sigma^f(T_1, T_2, u) \leq \epsilon.$$

For every natural number i put

$$v_i(t) = u(t + t_i),\ t \in [-t_i, \infty). \tag{3.81}$$

By (3.81), property (a), (3.79), and the continuity of U^f (see Proposition 3.8), for each integer $k \geq 1$, the sequence $\{I^f(-k, k, v_i) : i \text{ is an integer and } t_i \geq k\}$ is bounded. Proposition 3.5 implies that there exist a subsequence $\{v_{i_q}\}_{q=1}^{\infty}$ and an a.c. function $v : R^1 \to R^n$ such that for each integer $k \geq 1$, we have

$$v_{i_q}(t) \to v(t) \text{ as } q \to \infty \text{ uniformly on } [-k, k],$$

$$I^f(-k, k, v) \leq \liminf_{q \to \infty} I^f(-k, k, v_{i_q}). \tag{3.82}$$

It follows from (3.80)–(3.82) that $v(0) = x$ and that $v(t) \in H(f)$ for all $t \in R^1$. By property (b), (3.81), and (3.82), $\sigma^f(-k, k, v) = 0$ for all natural numbers k. This completes the proof of Proposition 3.10.

Proof of Proposition 3.11: For each pair of points $x, u \in R^n$ denote by $A(x, u)$ the matrix

$$((\partial^2 f / \partial u_i \partial u_j)(x, u))_{i,j=1}^n,$$

by $B(x,u)$ the matrix $((\partial^2 f/\partial u_i \partial x_j)(x,u))_{i,j=1}^n$, and by $C(x,u)$ the vector $(\partial f/\partial x_i)(x,u))_{i=1}^n$. Since $A(x,u)$ is positive definite for all $(x,u) \in R^n \times R^n$ there exists $A^{-1}(x,u)$, $(x,u) \in R^n \times R^n$. Define

$$v_3(t) = v_1(t),\ t \in [T_1, s],\ v_3(t) = v_2(t),\ t \in (s, T_2]. \tag{3.83}$$

Evidently, the a.c. function v_3 is well defined and

$$v_3(s) = v_1(s) = v_2(s). \tag{3.84}$$

It is clear that

$$\sigma^f(T_1, T_2, v_3) = \sigma^f(T_1, s, v_3) + \sigma^f(s, T_2, v_3) = 0. \tag{3.85}$$

By (3.77) and (3.85),

$$I^f(T_1, T_2, v_i) = U^f(0, T_2 - T_1, v_i(T_1), v_i(T_2)),\ i = 1, 2, 3. \tag{3.86}$$

It follows from (3.86) and Proposition 3.19 that for $i = 1, 2, 3$, the function $v_i \in C^2([T_1, T_2]; R^n)$ is a solution of the differential equation (3.78) on $[T_1, T_2]$ which can be written as

$$\begin{aligned} w''(t) &= -(A(w(t), w'(t)))^{-1} B(w(t), w'(t)) w'(t) \\ &\quad + (A(w(t), w'(t)))^{-1} C(w(t), w'(t)),\ t \in [T_1, T_2]. \end{aligned} \tag{3.87}$$

Since v_p, $p = 1, 2, 3$ are solutions of (3.87) it follows from (3.84), (3.83), and the inclusion $A, B, C \in C^1$ that $v_1(t) = v_2(t) = v_3(t)$ for all $t \in [T_1, T_2]$. This completes the proof of Proposition 3.11.

Corollary 3.23. *Assume that an integrand $f \in \mathcal{N}$ possesses ATP, $v : R^1 \to R^n$ is an a.c. function satisfying $v(0) \in H(f)$ and that $\sigma^f(-T, T, v) = 0$ for all positive numbers T. Then $v(R^1) \subset H(f)$.*

Proof of Proposition 3.13: It is easy to see that if $v \in \sigma(f)$, then the function v is c-optimal with respect to f. Assume that the function $v : R^1 \to H(f)$ is c-optimal with respect to f. We claim that $v \in \sigma(f)$. Assume the contrary. Then there exists a positive number T satisfying

$$\lambda := \sigma^f(-T, T, v) > 0.$$

Proposition 3.21 implies that there exists a real number $q \geq 8$ such that for each pair of points $x, y \in H(f)$ there exists an a.c. function $h : [0, q] \to R^n$ which satisfies

$$h(0) = x,\ h(q) = y,\ \sigma^f(0, q, h) \leq \lambda/8.$$

It follows from Proposition 3.10 that there exists a function $w \in \sigma(f)$ satisfying $w(T+q) = v(T+q)$. In view of the choice of q there exists an a.c. function $h : [-T, -T+q] \to R^n$ which satisfies

$$h(-T) = v(-T),\ h(-T+q) = w(-T+q),\ \sigma^f(-T, q-T, h) \leq \lambda/8.$$

Define

$$v_0(t) = v(t),\ t \in (-\infty, -T],\ v_0(t) = h(t),\ t \in (-T, -T+q],$$

$$v_0(t) = w(t),\ t \in (q-T, \infty).$$

It is clear that the a.c. function v_0 is well defined and

$$v_0(-T) = v(-T),\ v_0(T+q) = w(T+q) = v(T+q),$$

$$\sigma^f(-T, T+q, v_0) = \sigma^f(-T, q-T, v_0) + \sigma^f(q-T, q+T, v_0),$$
$$= \sigma^f(-T, q-T, h) + \sigma^f(q-T, T+q, w) \geq \lambda/8.$$

On the other hand

$$\sigma^f(-T, q+T, v) \geq \sigma^f(-T, T, v) \geq \lambda$$

and

$$I^f(-T, q+T, v_0) - I^f(-T, q+T, v)$$
$$= \sigma^f(-T, q+T, v_0) - \sigma^f(-T, q+T, v) \leq -(3/4)\lambda < 0,$$

a contradiction. The contradiction we have reached completes the proof of Proposition 3.13.

3.9 Proof of Theorem 3.14

Let $f \in \mathcal{N}$ possess ATP. In the proof of Theorem 3.14 we use the following several auxiliary results.

Lemma 3.24. *Let ϵ be a positive number. Then there exists a positive number L such that for each a.c. function $v : R^1 \to H(f)$ which satisfies $\sigma^f(-T, T, v) = 0$ for all positive numbers T, the inequality*

$$dist(H(f), v([-L, L])\} \leq \epsilon$$

holds.

Proof. Assume the contrary. Then there exists a sequence of a.c. functions $v_i : R^1 \to H(f)$, $i = 1, 2, \dots$ such that for each natural number i,

$$\sigma^f(-T, T, v_i) = 0 \text{ for all } T > 0, \tag{3.88}$$

$$\mathrm{dist}(H(f), v_i([-i, i])) \geq \epsilon. \tag{3.89}$$

By (3.88), the boundedness of $H(f)$, continuity of π^f, and (3.75), for each integer $k \geq 1$, the sequence $\{I^f(-k, k, v_i)\}_{i=1}^{\infty}$ is bounded. When combined with Proposition 3.5, this implies that there exist a subsequence $\{v_{i_q}\}_{q=1}^{\infty}$ and an a.c. function $v : R^1 \to R^n$ such that for each integer $k \geq 1$, we have

$$v_{i_q}(t) \to v(t) \text{ as } q \to \infty \text{ uniformly on } [-k, k], \tag{3.90}$$

$$I^f(-k, k, v) \leq \liminf_{q \to \infty} I^f(-k, k, v_{i_q}). \tag{3.91}$$

In view of (3.90), $v(R^1) \subset H(f)$. It follows from (3.91), (3.88), (3.75), (3.90), and the continuity of π^f that for each natural number k, we have

$$\sigma^f(-k, k, v) = I^f(-k, k, v) - 2k\mu(f) - \pi^f(v(-k)) + \pi^f(v(k))$$

$$\leq \liminf_{q \to \infty} [I^f(-k, k, v_{i_q}) - 2k\mu(f) - \pi^f(v_{i_q}(-k)) + \pi^f(v_{i_q}(k))]$$

$$\leq \liminf_{q \to \infty} \sigma^f(-k, k, v_{i_q}) = 0.$$

Hence

$$\sigma^f(-k, k, v) = 0 \text{ for all integers } k \geq 1. \tag{3.92}$$

Since the integrand f possesses ATP there exists a positive number L such that

$$\mathrm{dist}(H(f), v([-L, L])) \leq \epsilon/2. \tag{3.93}$$

By (3.90) and (3.93), there exists an integer $q_0 \geq 1$ satisfying

$$\mathrm{dist}(H(f), v_{i_q}([-L, L])) \leq (3/4)\epsilon$$

for all integers $q \geq q_0$. This contradicts (3.89). The contradiction we have reached completes the proof of Lemma 3.24.

Lemma 3.25. *Assume that* $\epsilon, L > 0$. *Then there exists a positive number* δ *such that for each pair of functions* $v_1, v_2 \in \sigma(f)$ *which satisfy* $|v_1(0) - v_2(0)| \leq \delta$, *the inequality*

$$|v_1(t) - v_2(t)| \leq \epsilon, \; t \in [-L, L]$$

holds.

Proof. Assume the contrary. Then for each integer $i \geq 1$ there exist a pair of functions

$$v_{1i}, v_{2i} \in \sigma(f) \tag{3.94}$$

which satisfy

$$|v_{1i}(0) - v_{2i}(0)| \le 1/i, \tag{3.95}$$

$$\sup\{|v_{1i}(t) - v_{2i}(t)| : t \in [-L, L]\} \ge \epsilon. \tag{3.96}$$

By (3.94) and the definition of $\sigma(f)$, for all natural numbers i, we have

$$v_{1i}(R^1),\ v_{2i}(R^1) \subset H(f), \tag{3.97}$$

$$\sigma^f(-k, k, v_{1i}),\ \sigma^f(-k, k, v_{2i}) = 0 \text{ for any integer } k \ge 1. \tag{3.98}$$

By (3.97), the boundedness of $H(f)$, the continuity of π^f, and (3.75), for each natural number k, the sequences $\{I^f(-k, k, v_{1i})\}_{i=1}^\infty$, $\{I^f(-k, k, v_{2i})\}_{i=1}^\infty$ are bounded. Proposition 3.5 implies that there exist subsequences $\{v_{1i_q}\}_{q=1}^\infty$, $\{v_{2i_q}\}_{q=1}^\infty$ and a.c. functions $v_1, v_2 : R^1 \to R^n$ such that for each natural number k and j=1,2, we have

$$v_{ji_q}(t) \to v_j(t) \text{ as } q \to \infty \text{ uniformly in } [-k, k], \tag{3.99}$$

$$I^f(-k, k, v_j) \le \liminf_{q\to\infty} I^f(-k, k, v_{ji_q}). \tag{3.100}$$

It follows from (3.99) and (3.97) that

$$v_j(R^1) \subset H(f),\ j = 1, 2. \tag{3.101}$$

By (3.75), (3.100), (3.99), the continuity of π^f, and (3.98), for each natural number k and $j = 1, 2$, we have

$$\begin{aligned} \sigma^f(-k, k, v_j) &= I^f(-k, k, v_j) - 2k\mu(f) - \pi^f(v_j(-k)) + \pi^f(v_j(k)) \\ &\le \liminf_{q\to\infty}[I^f(-k, k, v_{ji_q}) - 2k\mu(f) - \pi^f(v_{ji_q}(-k)) + \pi^f(v_{ji_q}(k))] \\ &= \lim_{q\to\infty} \sigma^f(-k, k, v_{ji_q}) = 0. \end{aligned}$$

Therefore

$$\sigma^f(-k, k, v_1),\ \sigma^f(-k, k, v_2) = 0 \text{ for all natural numbers } k. \tag{3.102}$$

In view of (3.95) and (3.99),

$$v_1(0) = v_2(0).$$

It follows from this equality, (3.102), (3.101), and Corollary 3.12 that

$$v_1(t) = v_2(t) \text{ for all } t \in R^1.$$

When combined with (3.99) this implies that there exists an integer $p \ge 1$ such that for $j = 1, 2$, we have

$$|v_{ji_p}(t) - v_1(t)| \le \epsilon/4,\ t \in [-L, L].$$

This inequality implies that

$$|v_{1i_p}(t) - v_{2i_p}(t)| \le \epsilon/2, \; t \in [-L, L],$$

a contradiction (see (3.96)). The contradiction we have reached completes the proof of Lemma 3.25.

Lemma 3.26. *Let ϵ be a positive number. Then there exists a positive number δ such that for each real number $T \ge 4$ and each a.c. function $v : [0, T] \to R^n$ which satisfies*

$$d(v(0), H(f)), \; d(v(T), H(f)) \le \delta, \tag{3.103}$$

$$\sigma^f(0, T, v) \le \delta, \tag{3.104}$$

the inequality

$$d(v(t), H(f)) \le \epsilon \text{ for all } t \in [0, T] \tag{3.105}$$

holds.

Proof. Theorem 3.9 implies that there exists a pair of real numbers $\delta_0 \in (0, 1)$ and $l_0 > 0$ such that for each real number $T \ge 2l_0$ and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$|v(0)|, |v(T)| \le \sup\{|z| : \; z \in H(f)\} + 4, \tag{3.106}$$

$$I^f(0, T, v) \le U^f(0, T, v(0), v(T)) + \delta_0, \tag{3.107}$$

we have

$$d(v(t), H(f)) \le \epsilon \text{ for all } t \in [l_0, T - l_0]. \tag{3.108}$$

Since the functions U^f, π^f are continuous and $H(f)$ is compact, there exists a real number

$$\delta \in (0, \delta_0) \tag{3.109}$$

such that for each $y_1, y_2, x_1, x_2 \in R^n$ which satisfy

$$d(y_i, H(f)), \; d(x_i, H(f)) \le 1, \; i = 1, 2,$$

$$|x_1 - x_2| \le \delta, \; |y_1 - y_2| \le \delta, \tag{3.110}$$

we have

$$|U^f(0, 1, x_1, y_1) - U^f(0, 1, x_2, y_2)|, \; |\pi^f(x_1) - \pi^f(x_2)| \le \delta_0/8. \tag{3.111}$$

Assume that a real number $T \ge 4$ and that an a.c. function $v : [0, T] \to R^n$ satisfies (3.103) and (3.104). We claim that (3.105) is true.

In view (3.103) there exists a pair of points $x_1, x_2 \in H(f)$ satisfying

$$|v(0) - x_1|, |v(T) - x_2| \le \delta. \tag{3.112}$$

Proposition 3.10 implies that there exists a pair of functions $u_1, u_2 \in \sigma(f)$ satisfying

$$u_1(0) = x_1, \ u_2(0) = x_2. \tag{3.113}$$

It follows from Proposition 3.20 that there exists an a.c. function $w : R^1 \to R^n$ which satisfies

$$w(t) = v(t), \ t \in [0, T], \tag{3.114}$$

$$w(T + 1 + t) = u_2(t + 1), \ t \in [0, \infty), \ w(t) = u_1(t), \ t \in (-\infty, -1],$$

$$I^f(T, T+1, w) = U^f(0, 1, v(T), u_2(1)), \ I^f(-1, 0, w) = U^f(0, 1, u_1(-1), v(0)).$$

Since $u_1, u_2 \in \sigma(f)$ we conclude that

$$u_1(t), \ u_2(t) \in H(f) \text{ for all } t \in R^1, \tag{3.115}$$

$$\sigma^f(-T, T, u_1) = \sigma^f(-T, T, u_2) = 0 \text{ for all } T > 0. \tag{3.116}$$

By the choice of δ (see (3.109)–(3.111)), (3.115), (3.113), and (3.12),

$$|U^f(0, 1, v(T), u_2(1)) - U^f(0, 1, u_2(0), u_2(1))| \le \delta_0/8,$$

$$|U^f(0, 1, u_1(-1), v(0)) - U^f(0, 1, u_1(-1), u_1(0))| \le \delta_0/8,$$

$$|\pi^f(v(T)) - \pi^f(u_2(0))| \le \delta_0/8, \ |\pi^f(v(0)) - \pi^f(u_1(0))| \le \delta_0/8. \tag{3.117}$$

By (3.75), (3.114), (3.117), and the inclusion $u_1, u_2 \in \sigma(f)$,

$$\begin{aligned}
\sigma^f(T, T + 1, w) &= I^f(T, T + 1, w) - \mu(f) - \pi^f(w(T)) + \pi^f(w(T + 1)) \\
&= -\mu(f) + U^f(0, 1, v(T), u_2(1)) - \pi^f(v(T)) + \pi^f(u_1(1)) \\
&\le -\mu(f) + U^f(0, 1, u_2(0), u_2(1)) + \delta_0/8 - \pi^f(u_2(0)) + \delta_0/8 + \pi^f(u_2(1)) \\
&\le I^f(0, 1, u_2) - \mu(f) - \pi^f(u_2(0)) + \pi^f(u_2(1)) + \delta_0/4 \\
&= \sigma^f(0, 1, u_2) + \delta_0/4 = \delta_0/4
\end{aligned}$$

and that

$$\begin{aligned}
\sigma^f(-1, 0, w) &= I^f(-1, 0, w) - \mu(f) - \pi^f(u_1(-1)) + \pi^f(v(0)) \\
&= -\mu(f) + U^f(0, 1, u_1(-1), v(0)) - \pi^f(u_1(-1)) + \pi^f(v(0)) \\
&\le -\mu(f) + U^f(0, 1, u_1(-1), u_1(0)) + \delta_0/8 - \pi^f(u_1(-1)) + \pi^f(u_1(0)) + \delta_0/8 \\
&\le -\mu(f) + \delta_0/4 + I^f(-1, 0, u_1) - \pi^f(u_1(-1)) + \pi^f(u_1(0)) \\
&= \sigma^f(-1, 0, u_1) + \delta_0/4 = \delta_0/4.
\end{aligned}$$

It follows from these relations, (3.116), and (3.114) that

$$\sigma^f(-\tau, \tau, w) \le \delta_0/2 \text{ for all } \tau > 0.$$

By the inequality above,

$$I^f(-\tau, \tau, w) \leq U^f(0, 2\tau, w(-\tau), w(\tau)) + \delta_0/2$$

for all positive numbers τ. It follows from this inequality, (3.114), (3.115), and the choice of δ_0, l_0 (see (3.106), (3.107)) that

$$d(w(t), H(f)) \leq \epsilon \text{ for all } t \in R^1.$$

When combined with (3.114) this implies (3.105). Lemma 3.26 is proved.

Lemma 3.27. *Let $\epsilon \in (0, 1)$. Then there exists a pair of real numbers $q \geq 8$ and $\delta > 0$ such that for each real number $T \geq q$ and each pair of points $x, y \in R^n$ satisfying $d(x, H(f)),\ d(y, H(f)) \leq \delta$, there exists an a.c. function $v : [0, T] \to R^n$ which satisfies*

$$v(0) = x,\ v(T) = y,\ \sigma^f(0, T, v) \leq \epsilon.$$

Proof. Proposition 3.21 implies that there exists a real number $q_0 \geq 8$ such that for each pair of points $h_1, h_2 \in H(f)$, there exists an a.c. function $v : [0, q_0] \to R^n$ which satisfies

$$v(0) = h_1,\ v(q_0) = h_2,\ \sigma^f(0, q_0, v) \leq \epsilon/8. \tag{3.118}$$

Since U^f, π^f are continuous and $H(f)$ is a compact there is a real number $\delta \in (0, 1/4)$ such that for each $x_1, x_2, y_1, y_2 \in R^n$ which satisfy

$$d(y_i, H(f)),\ d(x_i, H(f)) \leq 1,\ i = 1, 2, \tag{3.119}$$

$$|x_1 - x_2|,\ |y_1 - y_2| \leq \delta,$$

we have

$$|U^f(0, 1, x_1, y_1) - U^f(0, 1, x_2, y_2)| \leq \epsilon/16,\ |\pi^f(x_1) - \pi^f(x_2)| \leq \epsilon/16. \tag{3.120}$$

Fix a real number

$$q \geq q_0 + 2. \tag{3.121}$$

Assume that

$$T \geq q,\ x, y \in R^n,\ d(x, H(f)),\ d(y, H(f)) \leq \delta. \tag{3.122}$$

There exists a pair of points

$$x_1,\ y_1 \in H(f) \tag{3.123}$$

satisfying

$$|x - x_1|,\ |y - y_1| \leq \delta. \tag{3.124}$$

Proposition 3.10 and (3.123) imply that there exist $u_1, u_2 \in \sigma(f)$ satisfying

$$u_1(0) = x_1,\ u_2(0) = y_1. \tag{3.125}$$

It is clear that

$$u_1(R^1),\ u_2(R^1) \subset H(f), \tag{3.126}$$
$$\sigma^f(-i, i, u_j) = 0 \text{ for } j = 1, 2 \text{ and any natural number } i.$$

In view of the choice of q_0 (see (3.118)) and (3.126) there is an a.c. function $u_3 : [1, q_0 + 1] \to R^n$ which satisfies

$$u_3(1) = u_1(1) \in H(f),\ u_3(q_0 + 1) = u_2(q_0 - T + 1) \in H(f), \tag{3.127}$$
$$\sigma^f(1, q_0 + 1, u_3) \le \epsilon/8.$$

It follows from (3.127) and Proposition 3.20 that there exists an a.c. function $v : [0, T] \to R^n$ which satisfies

$$v(0) = x,\ v(t) = u_3(t),\ t \in [1, q_0 + 1],\ I^f(0, 1, v) = U^f(0, 1, x, u_3(1)),$$
$$v(t) = u_2(t - T),\ t \in (q_0 + 1, T - 1], \tag{3.128}$$
$$v(T) = y,\ I^f(T - 1, T, v) = U^f(0, 1, v(T - 1), y).$$

It is clear that the function v is well defined. In order to complete the proof of the lemma, it is sufficient to show that $\sigma^f(0, T, v) \le \epsilon$. By (3.128), (3.127), and (3.126),

$$\sigma^f(0, T, v) = \sigma^f(0, 1, v) + \sigma^f(1, q_0+1, v) + \sigma^f(q_0+1, T-1, v) + \sigma^f(T-1, T, v)$$
$$= \sigma^f(0, 1, v) + \sigma^f(1, q_0 + 1, u_3) + \sigma^f(q_0 + 1 - T, -1, u_2) + \sigma^f(T - 1, T, v)$$
$$\le \sigma^f(0, 1, v) + \epsilon/8 + \sigma^f(T - 1, T, v). \tag{3.129}$$

In view of the definition of δ (see (3.119)–(3.120)), (3.128), (3.127), (3.125), (3.126), and (3.124), we have

$$|U^f(0, 1, x, u_3(1)) - U^f(0, 1, u_1(0), u_1(1))| \le \epsilon/16,$$
$$|U^f(0, 1, v(T - 1), y) - U^f(0, 1, u_2(-1), u_2(0))| \le \epsilon/16,$$
$$|\pi^f(x) - \pi^f(u_1(0))| \le \epsilon/16,\ |\pi^f(y) - \pi^f(u_2(0))| \le \epsilon/16.$$

By these inequalities, (3.175), (3.128), (3.127), and (3.126),

$$\sigma^f(0, 1, v) = I^f(0, 1, v) - \mu(f) - \pi^f(v(0)) + \pi^f(v(1))$$
$$= U^f(0, 1, x, u_3(1)) - \mu(f) - \pi^f(x) + \pi^f(u_3(1))$$
$$\le U^f(0, 1, u_1(0), u_1(1)) + \epsilon/16 - \mu(f) - \pi^f(u_1(0)) + \epsilon/16 + \pi^f(u_1(1))$$
$$\le \sigma^f(0, 1, u_1) + \epsilon/8 = \epsilon/8,$$

$$\sigma^f(T-1,T,v) = I^f(T-1,T,v) - \mu(f) - \pi^f(v(T-1)) + \pi^f(v(T))$$
$$= U^f(0,1,v(T-1),y) - \mu(f) - \pi^f(v(T-1)) + \pi^f(y)$$
$$\leq U^f(0,1,u_2(-1),u_2(0)) + \epsilon/16 - \mu(f) - \pi^f(u_2(-1)) + \epsilon/16 + \pi^f(u_2(0))$$
$$\leq \sigma^f(-1,0,u_2) + \epsilon/8 = \epsilon/8.$$

It follows from these inequalities and (3.129) that $\sigma(0,T,v) \leq \epsilon$. Lemma 3.27 is proved.

Lemma 3.28. *Let ϵ be a positive number. Then there exists a pair of real numbers $l \geq 4$, $\delta > 0$ such that for each real number $T \geq l$ and each a.c. function $v : [0,T] \to R^n$ which satisfies*

$$d(v(0),H(f)) \leq \delta,\ d(v(T),H(f)) \leq \delta, \tag{3.130}$$

$$I^f(0,T,v) \leq U^f(0,T,v(0),v(T)) + \delta, \tag{3.131}$$

the following inequality holds:

$$\sigma^f(0,T,v) \leq \epsilon.$$

Proof. Lemma 3.27 implies that there exists a real number

$$\delta \in (0,\epsilon/8),\ l \geq 8 \tag{3.132}$$

such that for each real number $T \geq l$ and each pair of points $x, y \in R^n$ which satisfies

$$d(x,H(f)),\ d(y,H(f)) \leq \delta, \tag{3.133}$$

there exists an a.c. function $u : [0,T] \to R^n$ such that

$$u(0) = x,\ u(T) = y,\ \sigma^f(0,T,u) \leq \epsilon/8. \tag{3.134}$$

Assume that a real number $T \geq l$ and that an a.c. function $v : [0,T] \to R^n$ satisfies (3.130) and (3.131). By (3.130) and the choice of δ, l (see (3.132)–(3.134)), there exists an a.c. function $u : [0,T] \to R^n$ such that

$$u(0) = v(0),\ u(T) = v(T),\ \sigma^f(0,T,u) \leq \epsilon/8.$$

By (3.175), (3.131), and (3.132), we have

$$\sigma^f(0,T,v) = I^f(0,T,v) - T\mu(f) - \pi^f(v(0)) + \pi^f(v(T))$$

$$\leq \delta + I^f(0,T,u) - T\mu(f) - \pi^f(u(0)) + \pi^f(u(T)) \leq \epsilon/8 + \epsilon/8.$$

Lemma 3.28 is proved.

Lemma 3.29. *Assume that* ϵ, L *is a pair of positive numbers and that* $E \subset R^n$ *is a nonempty and bounded set. Then there exists a positive number* δ *such that for each pair of a.c. functions* $v_1, v_2 : [-L, L] \to R^n$ *which satisfies*

$$v_i(-L),\ v_i(L) \in E,\ i = 1, 2, \tag{3.135}$$

$$\sigma^f(-L, L, v_i) \le \delta,\ i = 1, 2, \tag{3.136}$$

$$|v_1(0) - v_2(0)| \le \delta, \tag{3.137}$$

the inequality

$$|v_1(t) - v_2(t)| \le \epsilon \text{ for all } t \in [-L, L]$$

holds.

Proof. Assume the contrary. Then for each natural number k there exist a.c. functions $v_{1k}, v_{2k} : [-L, L] \to R^n$ which satisfy

$$v_{ik}(-L),\ v_{ik}(L) \in E, \tag{3.138}$$

$$\sigma^f(-L, L, v_{ik}) \le 1/k,\ i = 1, 2, \tag{3.139}$$

$$|v_{1k}(0) - v_{2k}(0)| \le 1/k, \tag{3.140}$$

$$\sup\{|v_{1k}(t) - v_{2k}(t)| :\ t \in [-L, L]\} > \epsilon. \tag{3.141}$$

By (3.139), (3.138), the boundedness of E, and the continuity of U^f, the sequences $\{I^f(-L, L, v_{1k})\}_{k=1}^{\infty}$, $\{I^f(-L, L, v_{2k}\}_{k=1}^{\infty}$ are bounded. Proposition 3.5 implies that there exist subsequences $\{v_{1k_j}\}_{j=1}^{\infty}$, $\{v_{2k_j}\}_{j=1}^{\infty}$ and a.c. functions $v_1, v_2 : [-L, L] \to R^n$ such that for $i = 1, 2$, we have

$$v_{ik_j}(t) \to v_i(t) \text{ as } j \to \infty \text{ uniformly on } [-L, L], \tag{3.142}$$

$$I^f(-L, L, v_i) \le \liminf_{j\to\infty} I^f(-L, L, v_{ik_j}). \tag{3.143}$$

It follows from (3.75), (3.143), (3.142), the continuity of π^f, and (3.139) that for $i = 1, 2$, we have

$$\begin{aligned}
\sigma^f(-L, L, v_i) &= I^f(-L, L, v_i) - 2L\mu(f) - \pi^f(v_i(-L)) + \pi^f(v_i(L)) \\
&\le \liminf_{j\to\infty}[I^f(-L, L, v_{ik_j}) - 2L\mu(f) - \pi^f(v_{ik_j}(-L)) + \pi^f(v_{ik_j}(L))] \\
&= \liminf_{j\to\infty} \sigma^f(-L, L, v_{ik_j}) \le 0.
\end{aligned}$$

Therefore

$$\sigma^f(-L, L, v_i) = 0,\ i = 1, 2. \tag{3.144}$$

In view of (3.142) and (3.140), $v_1(0) = v_2(0)$. It follows from this equality and Proposition 3.11 that $v_1(t) = v_2(t)$ for all $t \in [-L, L]$. When combined with (3.142) this implies that for all sufficiently large integers $j \ge 1$, we have

$$\sup\{|v_{1k_j}(t) - v_{2k_j}(t)| : t \in [-L, L]\} \le \epsilon/2.$$

This contradicts (3.141). The contradiction we have reached completes the proof of Lemma 3.29.

Lemma 3.30. *Assume that $\epsilon > 0$ and $L > 0$. Then there exists a pair of real numbers $\delta > 0$ and $L_0 \ge 1 + 2L$ such that for each real number $T \ge L_0$, each a.c. function $v : [0, T] \to R^n$ satisfying*

$$d(v(0), H(f)),\ d(v(T), H(f)) \le \delta, \tag{3.145}$$

$$I^f(0, T, v) \le U^f(0, T, v(0), v(T)) + \delta, \tag{3.146}$$

each function $w \in \sigma(f)$, and each real number $\tau \in [L, T - L]$, there exists $s \in [0, L_0]$ such that

$$|v(\tau + t) - w(s + t)| \le \epsilon \text{ for all } t \in [-L, L]. \tag{3.147}$$

Proof. Lemma 3.29 implies that there exists a real number

$$\delta_1 \in (0, 1) \tag{3.148}$$

such that for each pair of a.c. functions $v_1, v_2 : [-L, L] \to R^n$ which satisfy

$$d(v_i(-L), H(f)),\ d(v_i(L), H(f)) \le 4,\ i = 1, 2 \tag{3.149}$$

$$\sigma^f(-L, L, v_i) \le 2\delta_1,\ i = 1, 2, \tag{3.150}$$

$$|v_1(0) - v_2(0)| \le 2\delta_1, \tag{3.151}$$

we have

$$|v_1(t) - v_2(t)| \le \epsilon,\ t \in [-L, L]. \tag{3.152}$$

In view of Lemma 3.24, there exists a real number $L_1 \ge 4$ satisfying

$$\mathrm{dist}(H(f), w([0, L_1])) \le \delta_1 \text{ for each } w \in \sigma(f). \tag{3.153}$$

Lemma 3.26 implies that there exists a real number $\delta_0 \in (0, \delta_1)$ such that for each real number $T \ge 4$ and each a.c. function $v : [0, T] \to R^n$ which satisfy

$$d(v(0), H(f)),\ d(v(T), H(f)) \le \delta_0, \tag{3.154}$$

$$\sigma^f(0, T, v) \le \delta_0,$$

we have

$$d(v(t), H(f)) \le \delta_1 \text{ for all } t \in [0, T]. \tag{3.155}$$

It follows from Lemma 3.28 that there exist

$$L_0 \ge L_1 + 2L,\ \delta \in (0, \delta_0) \tag{3.156}$$

such that for each real number $T \geq L_0$ and each a.c. function $v : [0, T] \to R^n$ which satisfy

$$d(v(0), H(f)) \leq \delta,\ d(v(T), H(f)) \leq \delta, \tag{3.157}$$

$$I^f(0, T, v) \leq U^f(0, T, v(0), v(T)) + \delta, \tag{3.158}$$

we have

$$\sigma^f(0, T, v) \leq \delta_0. \tag{3.159}$$

Assume that $T \geq L_0$, an a.c. function $v : [0, T] \to R^n$ satisfies (3.145) and (3.146), a function $w \in \sigma(f)$, and a real number $\tau \in [L, T - L]$. By (3.145), (3.146), and the choice of L_0, δ, inequality (3.159) holds. It follows from (3.145), (3.159), (3.156), and the choice of δ_0 that inequality (3.155) holds. Thus there exists a point $z_0 \in H(f)$ satisfying

$$|v(\tau) - z_0| \leq \delta_1. \tag{3.160}$$

In view of (3.153) there exists a real number $s \in [0, L_1]$ satisfying $|z_0 - w(s)| \leq \delta_1$. It follows from this inequality and (3.160) that $|v(\tau) - w(s)| \leq 2\delta_1$. By this inequality, (3.155), (3.159), and the choice of δ_1 (see (3.149)–(3.152)), inequality (3.147) holds. Lemma 3.30 is proved.

Proof of Theorem 3.14: We may assume that

$$\epsilon < 1,\ K > \sup\{|h| :\ h \in H(f)\} + 4,\ l \geq 1. \tag{3.161}$$

Proposition 2.7 implies that there exists a neighborhood $\mathcal{U}_1$ of the integrand f in the space $\mathcal{A}$ and a real number $K_1 > K$ such that for each integrand $g \in \mathcal{U}_1$, each pair of real numbers $T_1 \geq 0$, $T_2 \geq T_1 + 1$, and each a.c. function $v : [T_1, T_2] \to R^n$ which satisfies

$$|v(T_i)| \leq 2K + 4,\ i = 1, 2,\ I^g(T_1, T_2, v) \leq U^g(T_1, T_2, v(T_1), v(T_2)) + 4, \tag{3.162}$$

we have

$$|v(t)| \leq K_1,\ t \in [T_1, T_2]. \tag{3.163}$$

Lemma 3.30 implies that there exist $\delta \in (0, 1)$ and

$$l_1 \geq 1 + 2l \tag{3.164}$$

such that for each real number $T \geq l_1$ and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$d(v(0), H(f)),\ d(v(T), H(f)) \leq 8\delta, \tag{3.165}$$

$$I^f(0, T, v) \leq U^f(0, T, v(0), v(T)) + 8\delta, \tag{3.166}$$

each function $w \in \sigma(f)$, and each real number $\tau \in [l, T - l]$, there exists $s \in [0, l_1]$ which satisfies

$$|v(\tau + t) - w(s + t)| \leq \epsilon \text{ for all } t \in [-l, l]. \tag{3.167}$$

It follows from Proposition 3.22 that there exist a neighborhood $\mathcal{U}_2$ of the integrand f in the space $\mathcal{A}$ and a positive number N_0 such that for each integrand $g \in \mathcal{U}_2$ and each a.c. function $x : [0, N_0] \to R^n$ which satisfies

$$|x(0)|, |x(N_0)| \leq K_1, I^g(0, N_0, x) \leq U^g(0, N_0, x(0), x(N_0)) + 4, \tag{3.168}$$

there exists a real number $t \in [0, N_0]$ satisfying $d(x(t), H(f)) \leq \delta$.

Put

$$l_2 = 4l_1 + 4N_0 + 8. \tag{3.169}$$

Proposition 2.8 implies that there exists a neighborhood $\mathcal{U}_3$ of the integrand f in the space $\mathcal{A}$ such that for each integrand $g \in \mathcal{U}_3$, each real number $T \in [1, 10l_2+10]$, and each pair of points $x, y \in R^n$ which satisfies $|x|, |y| \leq K_1+4$, we have

$$|U^g(0, T, x, y) - U^f(0, T, x, y)| \leq \delta. \tag{3.170}$$

Since the function U^f is continuous there exists a positive number M_0 such that

$$|U^f(0, T, x, y)| \leq M_0 \tag{3.171}$$

for each real number $T \in [1, 10l_2 + 10]$ and each pair of points $x, y \in R^n$ satisfying $|x|, |y| \leq K_1 + 4$. It follows from Proposition 2.10 that there exists a neighborhood $\mathcal{U}_4$ of the integrand f in the space $\mathcal{A}$ such that for each integrand $g \in \mathcal{U}_4$, each pair of real numbers $T_1, T_2 \geq 0$ satisfying $T_2 - T_1 \in [1, 10l_2 + 16]$, and each a.c. function $y : [T_1, T_2] \to R^n$ which satisfies

$$\min\{I^g(T_1, T_2, y),\ I^f(T_1, T_2, y)\} \leq M_0 + 8, \tag{3.172}$$

we have

$$|I^g(T_1, T_2, y) - I^f(T_1, T_2, y)| \leq \delta. \tag{3.173}$$

Put

$$\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_3 \cap \mathcal{U}_4. \tag{3.174}$$

Assume that $g \in \mathcal{U}$, $T \geq 2l_2$ and an a.c. function $v : [0, T] \to R^n$ satisfies

$$|v(0)|, |v(T)| \leq K,\ I^g(0, T, v) \leq U^g(0, T, v(0), v(T)) + \delta. \tag{3.175}$$

By the choice of $\mathcal{U}_1$ (see (3.162), (3.163)), (3.174), and (3.175), we have

$$|v(t)| \leq K_1 \text{ for all } t \in [0, T]. \tag{3.176}$$

Assume that a pair of real numbers $S_1, S_2 \in [0, T]$ satisfies

$$d(v(S_i), H(f)) \leq \delta,\ i = 1, 2,\ S_2 - S_1 \in [l_1, l_2]. \tag{3.177}$$

We claim that for each real number $\tau \in [S_1 + l, S_2 - l]$ and each function $w \in \sigma(f)$, there exists $s \in [0, l_1]$ such that

$$|v(\tau + t) - w(s + t)| \le \epsilon, \ t \in [-l, l]. \tag{3.178}$$

It follows from (3.176), (3.177), (3.164), and the choice of $\mathcal{U}_3$ (see (3.170)) that

$$|U^f(S_1, S_2, v(S_1), v(S_2)) - U^g(S_1, S_2, v(S_1), v(S_2))| \le \delta. \tag{3.179}$$

By the choice of M_0 (see (3.171)), (3.177), (3.176), and (3.164), we have

$$|U^f(S_1, S_2, v(S_1), v(S_2))| \le M_0. \tag{3.180}$$

In view of (3.180), (3.175), and (3.179),

$$\begin{aligned} I^g(S_1, S_2, v) &\le U^g(S_1, S_2, v(S_1), v(S_2)) + \delta \\ &\le U^f(S_1, S_2, v(S_1), v(S_2)) + 2\delta \le M_0 + 2\delta. \end{aligned} \tag{3.181}$$

By the choice of $\mathcal{U}_4$ (see (3.172), (3.173)), (3.177), and (3.181), we have

$$|I^g(S_1, S_2, v) - I^f(S_1, S_2, v)| \le \delta. \tag{3.182}$$

It follows from (3.182), (3.175), and (3.179) that

$$\begin{aligned} I^f(S_1, S_2, v) &\le I^g(S_1, S_2, v) + \delta \le U^g(S_1, S_2, v(S_1), v(S_2)) + 2\delta \\ &\le U^f(S_1, S_2, v(S_1), v(S_2)) + 3\delta. \end{aligned} \tag{3.183}$$

By (3.183), (3.177), and the choice of δ (see (3.164)–(3.167)), for each function $w \in \sigma(f)$ and each real number $\tau \in [S_1 + l, S_2 - l]$, there exists a real number $s \in [0, l_1]$ such that (3.178) holds. Therefore we have shown that the following property holds:

(P1) For each pair of real numbers $S_1, S_2 \in [0, T]$ satisfying (3.177), each function $w \in \sigma(f)$, and each real number $\tau \in [S_1 + l, S_2 - l]$, there exists a real number $s \in [0, l_1]$ such that inequality (3.178) holds.

By (3.175), (3.176), and the definition of $\mathcal{U}_2$, N_0 (see (3.168)), for each real number $r_0 \in [0, T - 2N_0 - 2l_1]$, there exists a real number $r_1 \in [r_0 + l_1, r_0 + l_1 + N_0]$ which satisfies $d(v(r_1), H(f)) \le \delta$. This implies that there exists a finite sequence of numbers $\{S_i\}_{i=0}^{Q} \subset [0, T]$ such that

$$S_0 = 0, \ S_{i+1} - S_i \in [r_0 + l_1, r_0 + l_1 + N_0],$$

$$T - S_Q \le 2N_0 + 2l_1, \ d(v(S_i), H(f)) \le \delta, \ i = 1, \dots, Q.$$

The assertion of the theorem follows from these relations, (3.119), and property (P1).

3.10 Proof of Theorem 3.15

In order to prove Theorem 3.15 we need the following auxiliary results.

Proposition 3.31. *(Lemma 9.1 of [41], Chap. 3 of [51].) Assume that* $f \in \mathcal{A}$. *Then there exists a compact set* $H^* \subset R^n$ *which has the following properties:*
There exists an (f)*-good function* $u : [0, \infty) \to R^n$ *such that* $\Omega(u) = H^*$;
for each (f)*-good function* $v : [0, \infty) \to R^n$ *either* $\Omega(v) = H^*$ *or* $\Omega(v) \setminus H^* \neq \emptyset$.

Proposition 3.32. *(Lemmas 9.3 and 9.4 of [41], Chap. 3 of [51].) Let* $f \in \mathcal{A}$, H^* *be as guaranteed by Proposition 3.31 and let* $M > 0$ *satisfy* $\limsup_{t\to\infty} |v(t)| < M$ *for any* (f)*-good function* v *(see Proposition 2.6). Assume that* $\phi : R^n \to [0, \infty)$ *is a bounded continuous function such that*

$$H^* \subset \{x \in R^n : \ \phi(x) = 0\} \subset H^* \cup \{x \in R^n : \ |x| \geq M + 3\}.$$

For any $r \in (0, 1]$ *let*

$$f_r(x, u) = f(x, u) + r\phi(x), \ x, u \in R^n.$$

Then for any $r \in (0, 1]$, *we have* $f_r \in \mathcal{A}$, *and if* v *is and* (f_r)*-good function, then* $\Omega(v) = H^*$.

Proposition 3.33. *(Sect. 3, Chap. 2 of [5].) Let* Ω *be a closed subset of* R^q. *Then there exists a bounded nonnegative function* $\phi \in C^\infty(R^q)$ *such that* $\Omega = \{x \in R^q : \ \phi(x) = 0\}$ *and that for each sequence of nonnegative integers* $p_1, \dots, p_q$, *the function* $\partial^{|p|}\phi/\partial x_1^{p_1} \dots \partial x_q^{p_q} : R^q \to R^1$ *is bounded where* $|p| = \sum_{i=1}^q p_i$.

Proposition 3.34. *(Lemma 5.1 of [43], Chap. 5 of [51].) Assume that an integrand* $f \in \mathcal{N}$ *has ATP and that* $\epsilon > 0$. *Then there exists a neighborhood* $\mathcal{U}$ *of* f *in* $\mathcal{A}$ *such that for each* $g \in \mathcal{U}$ *and each* (g)*-good function* $v : [0, \infty) \to R^n$,

$$dist(\Omega(v), H(f)) \leq \epsilon.$$

Proof of Theorem 3.15: Propositions 3.31–3.33 imply that there exists a set $E \subset \mathcal{N}_k$ which is an everywhere dense subset of $\bar{\mathcal{N}}_k$ and such that each $f \in E$ possess ATP. It follows from Proposition 3.34 that for each integrand $f \in E$ and each natural number p, there exists an open neighborhood $\mathcal{U}(f, p)$ of the integrand f in the space$(\bar{\mathcal{N}}_k, \rho_k)$ such that for each integrand $g \in \mathcal{U}(f, p)$ and each (g)-good function $v : [0, \infty) \to R^n$, the inequality $\mathrm{dist}(H(f), \Omega(v)) \leq (4p)^{-1}$ holds.

Define

$$\mathcal{F}_{k0} = \cap_{p=1}^{\infty} \cup \{\mathcal{U}(f, p) : f \in E\}.$$

It is easy to see that $\mathcal{F}_{k0}$ is a countable intersection of open everywhere dense subsets of the space $\bar{\mathcal{N}}_k$.

Clearly, each integrand $f \in \mathcal{F}_{k0}$ possesses ATP. Theorem 3.15 is proved.

4

Convex Autonomous Problems

In this chapter we study the structure of approximate solutions of autonomous variational problems with convex integrands. The main goal is to study the structure of approximate solutions of the problems in the regions close to the endpoints of the domains. The results of this chapter provide the full description of the structure of solutions of the convex autonomous problems.

4.1 Preliminaries

Denote by $\mathrm{mes}(\Omega)$ the Lebesgue measure of a Lebesgue-measurable set $\Omega \subset R^1$.

Let $a > 0$ and let $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that $\psi(t) \to \infty$ as $t \to \infty$.

Denote by $|\cdot|$ the Euclidean norm in R^n and by $< \cdot, \cdot >$ the scalar product in R^n. Assume that a continuous function $f : R^n \times R^n \to R^1$ satisfies the following assumptions:

A(i) For each point $x \in R^n$ the function $f(x, \cdot) : R^n \to R^1$ is convex.

A(ii) $f(x,u) \geq \max\{\psi(|x|),\ \psi(|u|)|u|\} - a$ for each $(x,u) \in R^n \times R^n$.

A(iii) For each pair of positive numbers M, ϵ there exists a pair of positive numbers Γ, δ such that

$$|f(x_1,u_1) - f(x_2,u_2)| \leq \epsilon \max\{f(x_1,u_1),\ f(x_2,u_2)\}$$

for each $u_1, u_2, x_1, x_2 \in R^n$ which satisfy

$$|x_i| \leq M,\ |u_i| \geq \Gamma,\ i = 1,2 \text{ and } \max\{|x_1 - x_2|,\ |u_1 - u_2|\} \leq \delta.$$

Note that assumptions A(i)–A(iii) were introduced in Sect. 3.1 and that f belongs to the space of functions $\mathcal{A}$ which was also introduced in Sect. 3.1.

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In this chapter based on [52] we study the following two variational problems:

$$\int_0^T f(x(t), x'(t))dt \to \min \tag{P_1}$$

$$x : [0, T] \text{ is an a.c. function such that } x(0) = y \text{ and } x(T) = z$$

and

$$\int_0^T f(x(t), x'(t))dt \to \min \tag{P_2}$$

$$x : [0, T] \text{ is an a.c. function such that } x(0) = y,$$

where $T > 0$ and $y, z \in R^n$. When the function f is strictly convex, we will obtain a full description of the structure of approximate solutions of these two problems with sufficiently large T.

We study integral functionals

$$I^f(T_1, T_2, x) = \int_{T_1}^{T_2} f(x(t), x'(t))dt \tag{4.1}$$

where $-\infty < T_1 < T_2 < \infty$ and $x : [T_1, T_2] \to R^n$ is an absolutely continuous (a.c.) function.

For each pair of points $y, z \in R^n$ and each pair of real numbers T_1, T_2 satisfying $-\infty < T_1 < T_2 < \infty$, put

$$U^f(T_1, T_2, y, z) = \inf\{I^f(T_1, T_2, x) : \ x : [T_1, T_2] \to R^n$$

$$\text{is an a.c. function satisfying } x(T_1) = y, \ x(T_2) = z\}. \tag{4.2}$$

Clearly, the value $U^f(T_1, T_2, y, z) = U^f(0, T_2 - T_1, y, z)$ is finite for each pair of points $y, z \in R^n$ and each pair of real numbers T_1, T_2 satisfying $-\infty < T_1 < T_2 < \infty$.

For each positive number T and each point $y \in R^n$ put

$$\sigma^f(T, y) = \inf\{I^f(0, T, x) : \tag{4.3}$$

$$x : [0, T] \to R^n \text{ is an a.c. function satisfying } x(T_1) = y\},$$

$$\sigma^f(T) = \inf\{I^f(0, T, x) : \ x : [0, T] \to R^n \text{ is an a.c. function}\}. \tag{4.4}$$

Recall that an a.c. function $x : [0, \infty) \to R^n$ is called (f)-good if for any a.c. function $y : [0, \infty) \to R^n$, there exists a real number M_y depending on y such that $I^f(0, T, y) \geq M_y + I^f(0, T, x)$ for all real numbers $T \in (0, \infty)$.

Theorem 1.1 of [42] easily implies the following result.

Proposition 4.1. *For each point $z \in R^n$ there exists an (f)-good function $x : [0, \infty) \to R^n$ such that $x(0) = z$ and $I^f(0, T, x) = U^f(0, T, z, x(T))$ for all positive numbers T.*

For any a.c. function $x : [0,\infty) \to R^n$ put

$$J(x) = \liminf_{T\to\infty} T^{-1} I^f(0,T,x) \tag{4.5}$$

and set

$$\mu(f) = \inf\{J(x) :\ x : [0,\infty) \to R^n \text{ is an a.c. function}\}. \tag{4.6}$$

It is clear that the value $\mu(f)$ is finite and for every (f)-good function $x : [0,\infty) \to R^n$, we have

$$\mu(f) = J(x). \tag{4.7}$$

For every $x \in R^n$ put

$$\pi^f(x) = \inf\{\liminf_{T\to\infty}[I^f(0,T,v) - \mu(f)T] : \tag{4.8}$$

$$v : [0,\infty) \to R^n \text{ is an a.c. function satisfying } v(0) = x\}.$$

It was shown in [41] (see Theorems 8.1 and 8.2) that the function $\pi^f : R^n \to R^1$ is well defined and continuous.

For each point of pairs $x, y \in R^n$ and each positive number T, put

$$\theta_T^f(x,y) = U^f(0,T,x,y) - T\mu(f) - \pi^f(x) + \pi^f(y). \tag{4.9}$$

The function $\theta_T^f(x,y)$ $(T > 0,\ x, y \in R^n)$ possesses the following two properties (see Theorem 8.1 of [41]):

(P1) $(T,x,y) \to \theta_T^f(x,y)$, $(T,x,y) \in (0,\infty) \times R^n \times R^n$ is a nonnegative continuous function;

(P2) For every positive number T and every point $x \in R^n$ there exists a point $y \in R^n$ satisfying $\theta_T^f(x,y) = 0$.

In this chapter, we use the following notion known as the overtaking optimality criterion [12, 17, 37, 51].

We say that an a.c. function $v : [0,\infty) \to R^n$ is (f)-overtaking optimal if for each a.c. function $u : [0,\infty) \to R^n$ satisfying $u(0) = v(0)$, we have

$$\limsup_{T\to\infty}[\int_0^T f(v(t),v'(t))dt - \int_0^T f(u(t),u'(t))dt] \le 0.$$

The following optimality criterion for infinite horizon problems was introduced in the studies of the discrete Frenkel–Kontorova model related to dislocations in one-dimensional crystals [6, 38].

We say that an a.c. function $x : [0,\infty) \to R^n$ is (f)-minimal if

$$\sup\{|x(t)| :\ t \in [0,\infty)\} < \infty$$

and if for each positive number T and each a.c. function $y : [0,T] \to R^n$ satisfying $x(0) = y(0)$, $x(T) = y(T)$, we have $I^f(0,T,x) \le I^f(0,T,y)$.

We say that an a.c. function $x : (-\infty, 0] \to R^n$ is (f)-minimal if

$$\sup\{|x(t)| : t \in (-\infty, 0]\} < \infty$$

and if for each negative number T and each a.c. function $y : [T, 0] \to R^n$ satisfying $x(0) = y(0)$, $x(T) = y(T)$, we have $I^f(T, 0, x) \le I^f(T, 0, y)$.

Assume that $-\infty < T_1 < T_2 < \infty$ and that $v : [T_1, T_2] \to R^n$ is an a.c. function. Put

$$\Gamma^f(T_1, T_2, v) = I^f(T_1, T_2, v) - \mu(f)(T_2 - T_1) - \pi^f(v(T_1)) + \pi^f(v(T_2)). \quad (4.10)$$

It follows from (4.9) and (4.10) and property (P1) that

$$\Gamma^f(T_1, T_2, v) \ge 0 \text{ for each } T_1 \in R^1, \text{ each } T_1 > T_1,$$

$$\text{and each a.c. function } v : [T_1, T_2] \to R^n. \quad (4.11)$$

We say that an a.c. function $v : [0, \infty) \to R^n$ is (f)-perfect if $\Gamma^f(0, T, v) = 0$ for all positive numbers T.

In [41] (see Theorem 8.3) we proved the following result.

Proposition 4.2. *For every point $x \in R^n$ there exists an (f)-good and (f)-perfect function $v : [0, \infty) \to R^n$ such that $v(0) = x$.*

The next result follows from Proposition 2.1.

Proposition 4.3. *Any (f)-minimal function $x : [0, \infty) \to R^n$ is (f)-good.*

Let $x_* \in R^n$ be given. In this chapter we say that f has the turnpike property, or briefly, TP, and x_* is the turnpike of f if the following condition holds:

For each pair of positive numbers K, ϵ, there exists a pair of positive numbers δ, L such that for each pair of points $x, y \in R^n$ satisfying $|x|, |y| \le K$, each real number $T \ge 2L$, and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$v(0) = x, \ v(T) = y, \ I^f(0, T, v) \le U^f(0, T, x, y) + \delta,$$

the inequality $|v(t) - x_*| \le \epsilon$ holds for all real numbers $t \in [L, T - L]$.

The following result obtained in [52] will be proved in Sect. 4.3.

Theorem 4.4. *The function f possesses TP and $x_* \in R^n$ is the turnpike of f if and only if*

$$|v(t) - x_*| \to 0 \text{ as } t \to \infty$$

for each (f)-good function $v : [0, \infty) \to R^n$.

We will show that if f is a strictly convex function, then f possesses the turnpike property with a turnpike $x_* \in R^n$.

The main goal of this chapter based on [52] is to study the structure of approximate solutions of the problems (P_1) and (P_2) in the regions $[0, L]$ and $[T - L, T]$ (see the definition of the turnpike property).

Assume that f is a strictly convex function. For each point $z \in R^n$ we will construct (f)-minimal functions $\mathbf{z}^+ : [0, \infty) \to R^n$ and $\mathbf{z}^- : (-\infty, 0] \to R^n$ which are solutions of certain infinite horizon variational problems. We will also define two points $x_f, x_g \in R^n$ which are solutions of certain minimization problems. We will show that for each pair of positive numbers K, ϵ, there exists an integer $L \geq 1$ such that for each real number $T \geq 2L$, each pair of points $y, z \in R^n$ satisfying $|y|, |z| \leq K$, and each a.c. function $v : [0, T] \to R^n$, the following properties hold:

If the function v is a solution of the problem (P_1), then

$$|v(t) - x_*| \leq \epsilon \text{ for all } t \in [L, T - L],$$

$$|v(t) - \mathbf{y}^+(t)| \leq \epsilon \text{ for all } t \in [0, L],$$

$$|v(T - t) - \mathbf{z}^-(-t)| \leq \epsilon \text{ for all } t \in [0, L];$$

if the function v is a solution of the problem (P_2), then

$$|v(t) - x_*| \leq \epsilon \text{ for all } t \in [L, T - L],$$

$$|v(t) - \mathbf{y}^+(t)| \leq \epsilon \text{ for all } t \in [0, L],$$

$$|v(T - t) - \mathbf{x}_g^-(-t)| \leq \epsilon \text{ for all } t \in [0, L].$$

Thus our results provide the full description of the structure of solutions of the problems (P_1) and (P_2).

4.2 Turnpike Results

We use the notation and definitions introduced in Sect. 4.1. Let $f : R^n \times R^n \to R^1$ be a convex continuous function which satisfies A(i)–A(iii). We assume that f is a strictly convex function. It means that

$$f(\alpha(x_1, y_1) + (1 - \alpha)(x_2, y_2)) < \alpha f(x_1, y_1) + (1 - \alpha) f(x_2, y_2)$$

for each real number $\alpha \in (0, 1)$ and each pair of points $(x_1, y_1), (x_2, y_2) \in R^n \times R^n$ such that $(x_1, y_1) \neq (x_2, y_2)$.

Since the function f is continuous and strictly convex and satisfies A(ii), there exists a unique point $\bar{x} \in R^n$ which satisfies

$$f(\bar{x}, 0) = \inf\{f(z, 0) : z \in R^n\}. \tag{4.12}$$

It is a well-known fact of convex analysis that (4.12) implies the existence of a vector $\bar{l} \in R^n$ which satisfies

$$f(x,y) \geq f(\bar{x},0)+ < \bar{l}, y >, \ (x,y) \in R^n \times R^n. \tag{4.13}$$

In Sect. 4.4 we will prove the following two results.

Proposition 4.5. $\mu(f) = f(\bar{x},0)$.

Proposition 4.6. *If an a.c. function* $x : [0,\infty) \to R^n$ *is* (f)*-good, then* $|x(t) - \bar{x}| \to 0$ *as* $t \to \infty$.

Proposition 4.6 and Theorem 4.4 imply the following result.

Theorem 4.7. *The function* f *possesses TP and* $\bar{x}$ *is the turnpike of* f.

The following two results will also be established in Sect. 4.4.

Theorem 4.8. *Assume that* $x : [0,\infty) \to R^n$ *is an a.c. function. Then the following properties are equivalent, the function* x *is* (f)*-minimal; the function* x *is* (f)*-perfect and bounded, and the function* x *is* (f)*-overtaking optimal.*

Theorem 4.9. *For any point* $z \in R^n$ *there exists a unique* (f)*-overtaking optimal function* $v : [0,\infty) \to R^n$ *such that* $v(0) = z$.

Theorem 4.7 and Theorem 1.3 of [42] (see also Proposition 4.14) easily imply the following result.

Theorem 4.10. *Let* K, ϵ *be a pair of positive numbers. Then there exists a pair of positive numbers* δ, L *such that for each point* $x \in R^n$ *satisfying* $|x| \leq K$*, each real number* $T \geq 2L$*, and each a.c. function* $v : [0,T] \to R^n$ *which satisfies*

$$v(0) = x, \ I^f(0,T,v) \leq \sigma^f(T,x) + \delta,$$

the inequality $|v(t) - \bar{x}| \leq \epsilon$ *holds for all* $t \in [L, T-L]$.

Theorems 4.8 and 4.9 imply that for each point $x \in K$, there exists a unique (f)-minimal function $\mathbf{x}^+$ which satisfies $\mathbf{x}^+(0) = x$.

If $y \in R^n$ (respectively, $z \in R^n$), then the corresponding (f)-minimal function from y (respectively, from z) will be denoted by $\mathbf{y}^+$ (respectively, $\mathbf{z}^+$).

Put

$$g(x,y) = f(x,-y) \text{ for all } x,y \in R^n. \tag{4.14}$$

Evidently, g is also a continuous strictly convex function, A(i)–A(iii) hold for g, and that all the results stated for f are also true for g.

Note that an a.c. function $v : [0,\infty) \to R^n$ is (g)-minimal if and only if the function $t \to v(-t)$, $t \in (-\infty, 0]$ is (f)-minimal. When combined with Theorems 4.8 and 4.9 this fact implies that for each point $x \in R^n$, there exists a unique (f)-minimal function $\mathbf{x}^- : (-\infty, 0] \to R^n$ such that $\mathbf{x}^-(0) = x$. If $y \in R^n$ (respectively, $z \in R^n$), then $\mathbf{y}^-$ (respectively, $\mathbf{z}^-$) is an (f)-minimal function defined on $(-\infty, 0]$ which satisfies $\mathbf{y}^-(0) = y$ (respectively, $\mathbf{z}^-(0) = z$).

In Sect. 4.5 we will also prove the following useful auxiliary result.

Proposition 4.11. *There exists a unique pair of points $x_f, x_g \in R^n$ which satisfies*

$$\pi^f(x_f) \le \pi^f(z) \text{ for all } z \in R^n \text{ and } \pi^g(x_g) \le \pi^g(z) \text{ for all } z \in R^n.$$

The following two theorems established in [52] are the main results of this chapter. They describe the structure of approximate solutions of the problems (P_1) and (P_2) in the regions $[0, L]$ and $[T - L, T]$ (see the definition of the turnpike property). When combined with Theorem 4.7 these results provide the full description of the structure of approximate solutions of the problems (P_1) and (P_2) for sufficiently large positive numbers T.

Theorem 4.12. *Let ϵ and M be positive numbers. Then the following assertions hold:*

1. *Let L_1 be a natural number. Then there exist a pair of real numbers $\tau \ge 2L_1$ and $\delta_1 \in (0, 1)$ such that for each real number $T \ge 2\tau_1$, each pair of points $y, z \in R^n$ satisfying $|y|, |z| \le M$, and each a.c. function $v : [0, T] \to R^n$ which satisfies*

$$v(0) = y, \; v(T) = z, \; I^f(0, T, v) \le U^f(0, T, y, z) + \delta_1,$$

the following inequalities hold:

$$|v(t) - \mathbf{y}^+(t)| \le \epsilon \text{ for all } t \in [0, L_1]$$

$$\text{and } |v(T - t) - \mathbf{z}^-(-t)| \le \epsilon \text{ for all } t \in [0, L_1].$$

2. *There exists an integer $L_2 \ge 1$ such that for each pair of points $y, z \in R^n$ satisfying $|y|, |z| \le M$ and each real number $T \ge L_2$, the following inequality holds:*

$$|U^f(0, T, y, z) - T\mu(f) - \pi^f(y) - \pi^g(z)| \le \epsilon.$$

Theorem 4.13. *Let ϵ and M be positive numbers. Then the following assertions hold:*

1. *Let L_0 be a natural number. Then there exists a pair of real numbers $\tau \ge 2L_0$ and $\delta_1 \in (0, 1)$ such that for each real number $T \ge 2\tau$, each point $y \in R^n$ satisfying $|y| \le M$, and each a.c. function $v : [0, T] \to R^n$ which satisfies*

$$v(0) = y, \; I^f(0, T, v) \le \sigma^f(T, y) + \delta_1,$$

the inequality

$$|v(t) - \mathbf{y}^{+}(t)| \le \epsilon \text{ for all } t \in [0, L_0]$$

holds.

2. *Let L_1 be a natural number. Then there exists a pair of real numbers $\tau \ge 2L_1$ and $\delta_0 \in (0,1)$ such that for each real number $T \ge 2\tau$, each point $y \in R^n$ satisfying $|y| \le M$, and each a.c. function $v : [0,T] \to R^n$ which satisfies*

$$v(0) = y,\ I^f(0,T,v) \le \sigma^f(T,y) + \delta_0,$$

the inequality

$$|v(T-t) - \mathbf{x}_g^{-}(-t)| \le \epsilon \text{ for all } t \in [0, L_1]$$

holds.

3. *There exists an integer $L_2 \ge 1$ such that for each point $y \in R^n$ satisfying $|y| \le M$ and each real number $T \ge L_2$, the inequality*

$$|\sigma^f(T,y) - T\mu(f) - \pi^f(y) - \pi^g(x_g)| \le \epsilon$$

holds.

4.3 Proof of Theorem 4.4 and Auxiliary Results

Proof of Theorem 4.4: Theorem 2.2 implies that if f possesses TP and $x_* \in R^n$ is its turnpike, then $|v(t) - x_*| \to 0$ as $t \to \infty$ for any (f)-good function v.

Let $x_* \in R^n$ be given and let $|v(t) - x_*| \to 0$ as $t \to \infty$ for any (f)-good function v. Then it follows from Theorems 2.2 and 2.4 of [41] (see also Chap. 3 of [51]) that the integrand f possesses TP with the turnpike x_*.

Proposition 4.14. *(Theorem 1.3 of [42].) Assume that $M_1, M_2, c > 0$. Then there exists a positive number S such that for each real number $T \ge c$, the following properties hold:*

(i) For each pair of points $x, y \in R^n$ satisfying $|x|, |y| \le M_1$ and each a.c. function $v : [0,T] \to R^n$ which satisfies

$$v(0) = x,\ v(T) = y,\ I^f(0,T,v) \le U^f(0,T,x,y) + M_2,$$

the inequality

$$|v(t)| \le S \text{ for all } t \in [0,T] \tag{4.15}$$

holds.

(ii) For each point $x \in R^n$ satisfying $|x| \le M_1$ and each a.c. function $v : [0,T] \to R^n$ satisfying $v(0) = x$ and $I^f(0,T,v) \le \sigma^f(T,x) + M_2$, inequality (4.15) is true.

Put

$$L(x, y) = f(x, y) - f(\bar{x}, 0) - < \bar{l}, y >, \ (x, y) \in R^n \times R^n. \tag{4.16}$$

By (4.16) and (4.13) and strict convexity of f,

$$L(x, y) \geq 0 \text{ for all } (x, y) \in R^n \times R^n, \tag{4.17}$$

and for all points $(x, y) \in R^n \times R^n$, we have

$$L(x, y) = 0 \text{ if and only if } x = \bar{x}, \ y = 0.$$

Lemma 4.15. *Let ϵ be a positive number. Then there exists a positive number δ such that for each point $(x, y) \in R^n \times R^n$ satisfying $|x - \bar{x}| \geq \epsilon$, the inequality $L(x, y) \geq \delta$ is true.*

Proof. Assume the contrary. Then there exists a sequence $\{(x_i, y_i)\}_{i=1}^{\infty} \subset R^n \times R^n$ such that

$$|x_i - \bar{x}| \geq \epsilon \text{ for all integers } i \geq 1 \text{ and } \lim_{i \to \infty} L(x_i, y_i) = 0. \tag{4.18}$$

By (4.18), (4.16), A(ii), and the equality $\lim_{t \to \infty} \psi(t) = \infty$, the sequence $\{y_i\}_{i=1}^{\infty}$ is bounded. Since the sequence $\{y_i\}_{i=1}^{\infty}$ is bounded, relations (4.18) and (4.16), the assumption A(ii), and the relation $\lim_{t \to \infty} \psi(t) = \infty$ imply that the sequence $\{x_i\}_{i=1}^{\infty}$ is also bounded. Extracting a subsequence and re-indexing, we may assume without any loss of generality that there exist

$$x_* = \lim_{i \to \infty} x_i, \ y_* = \lim_{i \to \infty} y_i.$$

In view of these equalities and (4.18), we have $|x_* - \bar{x}| \geq \epsilon$ and $L(x_*, y_*) = 0$. These relations contradict (4.17). The contradiction we have reached completes the proof of Lemma 4.15.

Lemma 4.16. *Assume that $\epsilon > 0$ and $M > 0$. Then there exists a positive number δ such that for each $x_1, x_2, y_1, y_2 \in R^n$ satisfying $|x_1|, |x_2|, |y_1|, |y_2| \leq M$ and $|x_1 - x_2| \geq \epsilon$, the inequality*

$$f(2^{-1}(x_1 + x_2), 2^{-1}(y_1 + y_2)) \leq 2^{-1} f(x_1, y_1) + 2^{-1} f(x_2, y_2) - \delta$$

holds.

Proof. Assume the contrary. Then there exist sequences

$$\{(x_{1i}, y_{1i}\}_{i=1}^{\infty}, \ \{(x_{2i}, y_{2i})\}_{i=1}^{\infty} \subset R^n \times R^n$$

such that

$$|x_{1i}|, |x_{2i}|, |y_{1i}|, |y_{2i}| \leq M \text{ for all natural numbers } i,$$

$|x_{1i} - x_{2i}| \geq \epsilon$ for all natural numbers i, and

$$\lim_{i \to \infty} [f(2^{-1}(x_{1i} + x_{2i}), 2^{-1}(y_{1i} + y_{2i})) - 2^{-1} f(x_{1i}, y_{1i}) - 2^{-1} f(x_{2i}, y_{2i})] = 0.$$

Extracting a subsequence and re-indexing, if necessary, we may assume without any loss of generality that there exist

$$x_{j\infty} = \lim_{i \to \infty} x_{ji}, \; y_{j\infty} = \lim_{i \to \infty} y_{ji}$$

for $j = 1, 2$. Clearly,

$$|x_{1\infty} - x_{2\infty}| \geq \epsilon,$$

$$f(2^{-1}(x_{1\infty} + x_{2\infty}), 2^{-1}(y_{1\infty} + y_{2\infty})) = 2^{-1} f(x_{1\infty}, y_{1\infty}) + 2^{-1} f(x_{2\infty}, y_{2\infty}),$$

a contradiction. The contradiction we have reached completes the proof of Lemma 4.16.

Lemma 4.17. *Assume that $\epsilon \in (0, 1)$ and $M, c > 0$. Then there exists a positive number δ such that for each real number $T \geq c$ and each $x_1, x_2, y_1, y_2 \in R^n$ which satisfy*

$$|x_1|, |x_2|, |y_1|, |y_2| \leq M, \; |x_1 - x_2|, |y_1 - y_2| \leq \delta, \tag{4.19}$$

the inequality

$$|U^f(0, T, x_1, y_1) - U^f(0, T, x_2, y_2)| \leq \epsilon \tag{4.20}$$

holds.

Proof. Proposition 4.14 implies that there exists a positive number S_0 such that for each real number $T \geq c/4$ and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$|v(0)|, |v(T)| \leq M, \; I^f(0, T, v) \leq U^f(0, T, v(0), v(T)) + 1, \tag{4.21}$$

we have

$$|v(t)| \leq S_0 \text{ for all } t \in [0, T]. \tag{4.22}$$

It follows from Proposition 3.8 that there exists a positive number δ such that for each $x_1, x_2, y_1, y_2 \in R^n$ which satisfy

$$|x_1|, |x_2|, |y_1|, |y_2| \leq M + S_0 + 1, \; |x_1 - x_2|, |y_1 - y_2| \leq \delta, \tag{4.23}$$

we have

$$|U^f(0, c/4, x_1, y_1) - U^f(0, c/4, x_2, y_2)| \leq \epsilon/16. \tag{4.24}$$

Assume that $T \geq c$ and $x_1, x_2, y_1, y_2 \in R^n$ satisfy (4.19). In order to complete the proof of the lemma, it is sufficient to show that

$$U^f(0, T, x_2, y_2) \leq U^f(0, T, x_1, y_1) + \epsilon.$$

There exists an a.c. function $v_1 : [0, T] \to R^n$ which satisfies

$$v_1(0) = x_1,\ v_1(T) = y_1,\ I^f(0, T, v_1) \le U^f(0, T, x_1, y_1) + \epsilon/32. \tag{4.25}$$

There exists an a.c. function $v_2 : [0, c/4] \to R^n$, $v_3 : [T - c/4, T] \to R^n$ which satisfies

$$v_2(0) = x_2,\ v_2(c/4) = v_1(c/4), I^f(0, c/4, v_2) \le U^f(0, c/4, x_2, v_1(c/4)) + \epsilon/16, \tag{4.26}$$

$$v_3(T - c/4) = v_1(T - c/4),\ v_3(T) = y_2,$$

$$I^f(T - c/4, T, v_3) \le U^f(T - c/4, T, v_1(T - c/4), y_2) + \epsilon/16.$$

It follows from (4.25), (4.19), and the choice of S_0 (see (4.21) and (4.22)) that

$$|v_1(t)| \le S_0 \text{ for all } t \in [0, T]. \tag{4.27}$$

By the choice of δ (see (4.23) and (4.24)), (4.19), and (4.27), we have

$$|U^f(0, c/4, x_1, v_1(c/4)) - U^f(0, c/4, x_2, v_1(c/4))| \le \epsilon/16, \tag{4.28}$$

$$|U^f(T - c/4, T, v_1(T - c/4), y_1) - U^f(T - c/4, T, v_1(T - c/4), y_2)| \le \epsilon/16.$$

Put

$$u(t) = v_2(t),\ t \in [0, c/4],\ u(t) = v_1(t),\ t \in (c/4, T - c/4],$$

$$u(t) = v_3(t),\ t \in (T - c/4, T]. \tag{4.29}$$

It is easy to see that the a.c. function u is well defined on $[0, T]$. It follows from (4.29), (4.25), (4.26), and (4.28) that

$$U^f(0, T, x_2, y_2) \le I^f(0, T, u)$$
$$= I^f(0, c/4, v_2) + I^f(c/4, T - c/4, v_1) + I^f(T - c/4, T, v_3)$$
$$\le \epsilon/16 + U^f(0, c/4, x_2, v_1(c/4))$$
$$+[I^f(0, T, v_1) - I^f(0, c/4, v_1) - I^f(T - c/4, T, v_1)] + \epsilon/16$$
$$+U^f(T-c/4, T, v_1(T-c/4), y_2) \le U^f(0, c/4, x_2, v_1(c/4)) - U^f(0, c, x_1, v_1(c/4))$$
$$+U^f(T - c/4, T, v_1(T - c/4), y_2) - U^f(T - c/4, T, v_1(T - c/4), y_1)$$
$$+U^f(0, T, x_1, y_1) + \epsilon/32 + \epsilon/8 \le \epsilon/16 + \epsilon/16 + \epsilon/32 + \epsilon/8 + U^f(0, T, x_1, y_1).$$

This completes the proof of Lemma 4.17.

4.4 Proofs of Propositions 4.5 and 4.6 and Theorems 4.8 and 4.9

Proof of Proposition 4.5: It is easy to see that $\mu(f) \le f(\bar{x}, 0)$. Assume that $x : [0, \infty) \to R^n$ is an (f)-good function. It follows from (4.7) and (4.5) that

$$\mu(f) = \liminf_{T \to \infty} T^{-1} I^f(0, T, x). \tag{4.30}$$

Since x is an (f)-good function Proposition 2.6 implies that $\sup\{|x(t)| : t \in [0, \infty)\} < \infty$. This inequality, (4.30), and (4.13) imply that

$$\mu(f) = \liminf_{T \to \infty} T^{-1} \int_0^T f(x(t), x'(t))dt$$

$$\ge \liminf_{T \to \infty} T^{-1} \int_0^T [f(\bar{x}, 0) + < \bar{l}, x'(t) >]dt$$

$$= \liminf_{T \to \infty} [f(\bar{x}, 0) + T^{-1} < \bar{l}, x(T) - x(0) >] = f(\bar{x}, 0).$$

Proposition 4.5 is proved.

Proof of Proposition 4.6: Assume that $x : [0, \infty) \to R^n$ is an (f)-good function. We claim that $|x(t) - \bar{x}| \to 0$ as $t \to \infty$. Assume the contrary. Then there exist a positive number ϵ and a strictly increasing sequence of numbers $T_{k+1} \ge 4 + T_k$, $k = 1, 2 \ldots$ such that

$$|x(T_k) - \bar{x}| \ge 2\epsilon \text{ for all integers } k \ge 1. \tag{4.31}$$

Proposition 2.6 implies that there exists a positive number M such that

$$|x(t)| \le M \text{ for all } t \in [0, \infty). \tag{4.32}$$

It follows from Propositions 4.2 and 4.5, (4.13), and (4.16) that

$$\infty > \sup\{|I^f(0, T, x) - Tf(\bar{x}, 0)| : T \in (0, \infty)\}$$

$$= \sup\{\int_0^T L(x(t), x'(t))dt + < \bar{l}, x(T) - x(0) >: T \in (0, \infty)\}.$$

In view of this relation and (4.32), we have

$$\sup\{\int_0^T L(x(t), x'(t))dt : T \in (0, \infty)\} < \infty. \tag{4.33}$$

Since x is an (f)-good function Propositions 2.1 and 3.8 and (4.32) imply that there exists a positive number M_0 such that

$$I^f(s, s + 1, x) \le M_0 \text{ for all } s \in [0, \infty). \tag{4.34}$$

It follows from (4.34) and Proposition 2.12 that there exists a real number $\delta \in (0, 8^{-1})$ such that for each natural number k and each real number $t \in [T_k, T_k + \delta]$, we have

$$|x(t) - x(T_k)| \leq \epsilon.$$

When combined with (4.31) this implies that

$$\text{if } t \in \cup_{k=1}^{\infty}[T_k, T_k + \delta], \text{ then } |x(t) - \bar{x}| \geq \epsilon. \tag{4.35}$$

Together with Lemma 4.15, (4.35) implies that there exists a positive number γ for which

$$L(x(t), x'(t)) \geq \gamma \text{ for } t \in \cup_{k=1}^{\infty}[T_k, T_k + \delta] \text{ (a.e.)}.$$

When combined with relations (4.15) and (4.33) this inequality implies that

$$\begin{aligned} \infty > \sup\{\int_0^T L(x(t), x'(t))dt : \ T \in (0, \infty)\} \\ \geq \sum_{k=1}^{\infty} \int_{T_k}^{T_k+\delta} L(x(t), x'(t))dt \\ \geq \sum_{k=1}^{\infty} \gamma\delta = \infty. \end{aligned}$$

The contradiction we have reached completes the proof of Proposition 4.6.

Proposition 4.18. *Assume that $v_1, v_2 : [0, \infty) \to R^n$ are (f)-overtaking optimal functions satisfying $v_1(0) = v_2(0)$. Then $v_1(t) = v_2(t)$ for all nonnegative real numbers t.*

Proof. Assume the contrary. Then there exists a positive number τ such that

$$v_1(\tau) \neq v_2(\tau). \tag{4.36}$$

It is easy to see that

$$\lim_{T \to \infty} [I^f(0, T, v_1) - I^f(0, T, v_2)] = 0. \tag{4.37}$$

Put

$$u(t) = 2^{-1}[v_1(t) + v_2(t))], \ t \in [0, \infty). \tag{4.38}$$

Clearly,

$$f(u(t), u'(t)) \leq 2^{-1}[f(v_1(t), v_1'(t)) + f(v_2(t), v_2'(t))] \text{ for } t \in [0, \infty) \text{ a.e.,} \tag{4.39}$$

u is an (f)-overtaking optimal function such that $u(t) = v_1(0)$ and

$$\lim_{T \to \infty} [I^f(0, T, v_1) - I^f(t, T, u)] = 0. \tag{4.40}$$

It follows from (4.36) that there exists a positive number ϵ such that

$$|v_1(\tau) - v_2(\tau)| > 4\epsilon. \tag{4.41}$$

Propositions 4.14 and 2.6 imply that there exists a positive number M_0 such that

$$|v_i(t)| \le M_0 \text{ for all } t \in [0, \infty) \text{ and } i = 1.2. \tag{4.42}$$

Since v_1, v_2 are (f)-overtaking optimal, relation (4.42) and Proposition 3.8 imply that the integrals $I^f(0, \tau + 1, v_i)$, $i = 1, 2$ are finite. When combined with Proposition 2.12 this implies that there exists a real number $\delta \in (0, \min\{\tau, 1\}/8)$ such that for $i = 1, 2$, we have

$$|v_i(\tau) - v_i(t)| \le \epsilon \text{ for all } t \in [\tau - \delta, \tau + \delta].$$

By this inequality and (4.41), we have

$$|v_1(t) - v_2(t)| \ge 2\epsilon \text{ for all } t \in [\tau - \delta, \tau + \delta]. \tag{4.43}$$

Fix a positive number M_1 such that

$$I^f(0, \tau + 1, v_i) \le M_1,\ i = 1, 2. \tag{4.44}$$

Clearly, there exists a real number $c_0 > 1$ such that

$$\psi(t) \ge 1 \text{ for all } t \ge c_0 \text{ and } c_0^{-1}(M_1 + a(\tau + 1)) < \delta/8. \tag{4.45}$$

By (4.44) and A(ii), for $i = 1, 2$, we have

$$M_1 \ge I^f(0, \tau + 1, v_i) \ge \int_0^{\tau+1} \psi(|v_i'(t)|)|v_i'(t)|dt - a(\tau + 1).$$

By this inequality and (4.45), for $i = 1, 2$, we have

$$M_1 + a(\tau + 1) \ge (\text{mes}\{t \in [0, \tau + 1] :\ |v_i'(t)|$$

$$\ge c_0\})c_0\psi(c_0) \ge c_0(\text{mes}\{t \in [0, \tau + 1] :\ |v_i'(t)| \ge c_0\},$$

$$\text{mes}(\{t \in [0, \tau + 1] :\ |v_i'(t)| \ge c_0\}) \le c_0^{-1}(M_1 + a(\tau + 1)) < \delta/8. \tag{4.46}$$

Put

$$\Omega = \{t \in [\tau - \delta, \tau + \delta] :\ |v_i'(t)| < c_0 \text{ for } i = 1, 2\}. \tag{4.47}$$

It follows from (4.47) and (4.16) that

$$\mathrm{mes}(\Omega) \geq (3/2)\delta. \tag{4.48}$$

By (4.47), (4.43), (4.42), and Lemma 4.16, there exists a positive number γ such that for almost every real number $t \in \Omega$, we have

$$\begin{aligned} & f(2^{-1}(v_1(t) + v_2(t)), 2^{-1}(v_1'(t) + v_2'(t))) \\ & \leq 2^{-1} f(v_1(t), v_1'(t)) + 2^{-1} f(v_2(t), v_2'(t)) - \gamma. \end{aligned} \tag{4.49}$$

By (4.38), (4.39), and (4.47)–(4.49) that for all real numbers $T > \tau + 1$, we have

$$\begin{aligned} & 2^{-1} I^f(0, T, v_1) + 2^{-1} I^f(0, T, v_2) - I^f(0, T, u) \\ & \geq 2^{-1} \int_\Omega f(v_1(t), v_1'(t))dt + 2^{-1} \int_\Omega f(v_2(t), v_2'(t))dt \\ & - \int_\Omega f(u(t), u'(t))dt \geq \gamma \mathrm{mes}(\Omega) \geq (3/2)\gamma\delta. \end{aligned}$$

When combined with (4.37) this implies that for all sufficiently large real numbers $T > \tau + 1$ the inequality

$$I^f(0, T, v_1) - I^f(0, T, u) \geq (3/4)\gamma\delta$$

holds. This contradicts the assumption that the function v_1 is (f)-overtaking optimal. The contradiction we have reached completes the proof of Proposition 4.18.

Proposition 4.19. *Assume that $v : [0, \infty) \to R^n$ is a bounded (f)-perfect function. Then v is (f)-overtaking optimal.*

Proof. Since v is (f)-perfect for each positive number T, we have

$$I^f(0, T, v) = T\mu(f) + \pi^f(v(0)) - \pi^f(v(T)). \tag{4.50}$$

Since the function v is bounded, equality (4.50) and Proposition 3.4 imply that v is an (f)-good function, by Proposition 4.6,

$$\bar{x} = \lim_{t\to\infty} v(t). \tag{4.51}$$

Assume that an a.c. function $u : [0, \infty) \to R^n$ satisfies $u(0) = v(0)$. We claim that

$$\limsup_{T\to\infty} [I^f(0, T, v) - I^f(0, T, u)] \leq 0. \tag{4.52}$$

By Proposition 3.4, we may assume that u is an (f)-good function. In view of Proposition 4.6, we have

$$\bar{x} = \lim_{t\to\infty} u(t). \tag{4.53}$$

Let $\epsilon > 0$ be given. We claim that for all sufficiently large positive numbers T, the inequality

$$I^f(0, T, v) \le I^f(0, T, u) + \epsilon$$

holds.

Fix a real number $\delta \in (0, 1)$ such that

$$\text{if } z \in R^n \text{ satisfies } |\bar{x} - z| \le \delta, \text{ then } |\pi^f(z) - \pi^f(\bar{x})| \le \epsilon/8. \tag{4.54}$$

It follows from (4.51) and (4.53) that there exists a number T_0 such that for each real number $t \ge T_0$, we have

$$|v(t) - \bar{x}|, \ |u(t) - \bar{x}| \le \delta. \tag{4.55}$$

Let $T \ge T_0$ be given. It follows from (4.55) and (4.54) that

$$|\pi^f(v(T)) - \pi^f(\bar{x})|, \ |\pi^f(u(T)) - \pi^f(\bar{x})| \le \epsilon/8. \tag{4.56}$$

By (4.50), (4.10), (4.11), the equality $v(0) = u(0)$, and (4.56), we have

$$\begin{aligned} &I^f(0, T, v) - I^f(0, T, u) \\ &\le T\mu(f) + \pi^f(v(0)) - \pi^f(v(T)) - [\mu(f)T + \pi^f(u(0)) - \pi^f(u(T))] \\ &= \pi^f(u(T)) - \pi^f(v(T)) \le \epsilon/4. \end{aligned}$$

Since ϵ is an arbitrary positive number we conclude that (4.52) holds. This completes the proof of Proposition 4.19.

Proposition 4.20. *Assume that $v : [0, \infty) \to R^n$ is an (f)-minimal function. Then v is (f)-perfect.*

Proof. Propositions 4.3 and 4.6 imply that the function v is (f)-good and that

$$\lim_{t \to \infty} |v(t) - \bar{x}| = 0. \tag{4.57}$$

It follows from Proposition 4.2 that there exists (f)-good and (f)-perfect function $u : [0, \infty) \to R^n$ satisfying $u(0) = v(0)$. By Proposition 4.6, we have

$$\lim_{t \to \infty} |u(t) - \bar{x}| = 0. \tag{4.58}$$

Let $\epsilon > 0$ be given. Fix a positive number $\delta < 1$ such that

$$\text{if } z \in R^n \text{ satisfies } |z - \bar{x}| \le \delta, \ \text{ then } |\pi^f(z) - \pi^f(\bar{x})| \le \epsilon/8; \tag{4.59}$$

$$\begin{gathered} \text{if } (z_1, z_2) \in R^n \times R^n \text{ satisfies } |(z_1, z_2) - (\bar{x}, 0)| \le 4\delta, \\ \text{then } |f(z_1, z_2) - f(\bar{x}, 0)| \le \epsilon/8. \end{gathered}$$

It follows from (4.57) and (4.58) that there exists a real number T_0 such that for all real numbers $t \ge T_0$, relation (4.55) holds.

Let $T \geq T_0$ be given. Put

$$w(t) = u(t),\ t \in [0, T],\ w(T+s) = u(T)+t[v(T+1)-u(T)],\ t \in (0, 1]. \quad (4.60)$$

By (4.60) and the equality $u(0) = v(0)$, the a.c. function $w : [0, T + 1] \to R^n$ satisfies

$$w(s) = v(s),\ s = 0, T + 1. \quad (4.61)$$

Hence

$$I^f(0, T + 1, v) \leq I^f(0, T + 1, w).$$

Since u is (f)-perfect, the inequality above, (4.10), (4.60), and (4.61) imply that

$$0 \leq I^f(0, T+1, w) - I^f(0, T+1, v) = \Gamma^f(0, T, w) + \mu(f)T + \pi^f(w(0)) - \pi^f(w(T))$$

$$+ \int_T^{T+1} f(w(t), v(T + 1) - u(T))dt$$

$$-[\Gamma^f(0, T + 1, v) + \mu(f)(T + 1) + \pi^f(v(0)) - \pi^f(v(T + 1))]$$

$$-\pi^f(u(T)) + \int_T^{T+1} f(u(T) + (t - T)(v(T + 1) - u(T)), (v(T + 1) - u(T)))dt$$

$$-\Gamma^f(0, T + 1, v) - \mu(f) + \pi^f(v(T + 1)),$$

$$\Gamma^f(0, T + 1, v) \leq \pi^f(v(T + 1)) - \pi^f(u(T)) - \mu(f)$$

$$+ \int_T^{T+1} f(u(T) + (t - T)(v(T + 1) - u(T)), v(T + 1) - u(T))dt. \quad (4.62)$$

In view of (4.55) and (4.59), we have

$$|\pi^f(v(T + 1)) - \pi^f(u(T))| \leq \epsilon/4. \quad (4.63)$$

It follows from (4.55) that for each real number $t \in [T, T + 1]$, we have

$$|(u(T) + (t - T)(v(T + 1) - u(T)),\ v(T + 1) - u(T)) - (\bar{x}, 0)|$$

$$\leq |u(T) + (t - T)(v(T + 1) - u(T)) - \bar{x}| + |v(T + 1) - u(T)| \leq \delta + 2\delta.$$

When combined with (4.59) and Proposition 4.5 this relation implies that for all real numbers $t \in [T, T + 1]$, we have

$$|f(u(T) + (t - T)(v(T + 1) - u(T)), v(T + 1) - u(T)) - \mu(f)| \leq \epsilon/8.$$

By this inequality, (4.63), and (4.62),

$$\Gamma^f(0, T + 1, v) \leq \epsilon/4 + \epsilon/8$$

for all real numbers $T \geq T_0$. Since ϵ is any positive number we conclude that the function v is (f)-perfect. This completes the proof of Proposition 4.20.

Proof of Theorem 4.9: Assume that $z \in R^n$. Proposition 4.2 implies that there exists an (f)-good and (f)-perfect function $v : [0, \infty) \to R^n$ which satisfies $v(0) = z$. By Proposition 2.6, the function v is bounded. By Proposition 4.19, the function v is (f)-overtaking optimal. The uniqueness of (f)-overtaking optimal function follows from Proposition 4.18. This completes the proof of Theorem 4.9.

Proof of Theorem 4.8: Assume that the function x is (f)-minimal. By definition, the function x is bounded. By Proposition 4.20, the function x is (f)-perfect. Assume that x is (f)-perfect and bounded. Proposition 4.19 implies that the function x is (f)-overtaking optimal. Assume that x is an (f)-overtaking optimal function. It is clear that x is (f)-minimal. Theorem 4.8 is proved.

4.5 Auxiliary Results and Proof of Proposition 4.11

Proposition 4.21. *Assume that $\epsilon \in (0,1)$, $M > 0$, and $L \geq 1$ is an integer. Then there exists a positive number δ such that for each pair of points $x, y \in R^n$ satisfying $|x|, |y| \leq M$ and $|x - y| \leq \delta$, the inequality $|\mathbf{x}^+(t) - \mathbf{y}^+(t)| \leq \epsilon$ is true for all real numbers $t \in [0, L]$.*

Proof. Assume the contrary. Then for each integer $k \geq 1$ there exists a pair of points $x_k, y_k \in R^n$ such that

$$|x_k|, |y_k| \leq M, \ |x_k - y_k| \leq 1/k, \tag{4.64}$$

$$\sup\{|(\mathbf{x}_k)^+(t) - (\mathbf{y}_k)^+(t)| : \ t \in [0, L]\} > \epsilon. \tag{4.65}$$

By (4.64) and Propositions 2.6 and 4.14, there exists a real number $S_0 > M$ such that

$$|(\mathbf{x}_k)^+(t)|, \ |(\mathbf{y}_k)^+(t)| \leq S_0 \text{ for all } t \in [0, \infty) \text{ and all integers } k \geq 1. \tag{4.66}$$

It follows from (4.66) and Proposition 3.8 that the sequences

$$\{I^f(0, N, (\mathbf{x}_k)^+)\}_{k=1}^{\infty}, \ \{I^f(0, N, (\mathbf{y}_k)^+)\}_{k=1}^{\infty}$$

are bounded for all natural numbers N. Then in view of Proposition 3.5, there exist a strictly increasing sequence of natural numbers $\{k_j\}_{j=1}^{\infty}$ and a.c. functions $x : [0, \infty) \to R^n$, $y : [0, \infty) \to R^n$ such that for each integer $N \geq 1$, we have

$$(\mathbf{x}_{k_j})^+(t) \to x(t), \ (\mathbf{y}_{k_j})^+(t) \to y(t) \text{ as } k \to \infty \text{ uniformly on } [0, N], \tag{4.67}$$

$$I^f(0,N,x) \le \liminf_{j\to\infty} I^f(0,N,(\mathbf{x}_{k_j})^+),\ \ I^f(0,N,y) \le \liminf_{j\to\infty} I^f(0,N,(\mathbf{y}_{k_j})^+).$$

By (4.64), (4.65), and (4.67),

$$\sup\{|x(t)-y(t)| :\ t \in [0,L]\} \ge \epsilon, \tag{4.68}$$

$$x(0) = y(0). \tag{4.69}$$

It follows from (4.67) and (4.65) that

$$|x(t)|,\ |y(t)| \le S_0 \text{ for all } t \in [0,\infty). \tag{4.70}$$

By (4.67) and Proposition 3.8, for each natural number N, we have

$$I^f(0,N,x) \le \liminf_{j\to\infty} U^f(0,N,(\mathbf{x}_{k_j})^+(0),(\mathbf{x}_{k_j})^+(N)) = U^f(0,N,x(0),x(N)),$$

$$I^f(0,N,y) \le \liminf_{j\to\infty} U^f(0,N,(\mathbf{y}_{k_j})^+(0),(\mathbf{y}_{k_j})^+(N)) = U^f(0,N,y(0),y(N)).$$

Hence x, y are (f)-minimal functions. It follows from Theorems 4.8 and 4.9 and (4.69) that $x(t) = y(t)$ for all real numbers $t \in [0,\infty)$. This contradicts (4.68). The contradiction we have reached completes the proof of Proposition 4.21.

Proposition 4.21 applied for the function g implies the following auxiliary result.

Proposition 4.22. *Assume that $\epsilon \in (0,1)$, $M > 0$, and $L \ge 1$ is an integer. Then there exists a positive number δ such that for each pair of points $x, y \in R^n$ satisfying $|x|, |y| \le M$ and $|x-y| \le \delta$, the inequality $|\mathbf{x}^-(t) - \mathbf{y}^-(t)| \le \epsilon$ is valid for all real numbers $t \in [-L, 0]$.*

Proposition 4.23. $\pi^f(\bar{x}) = 0.$

Proof. Evidently, $(\bar{\mathbf{x}})^+(t) = \bar{x}$ for all $t \in 0,\infty)$. Then

$$\pi^f(\bar{x}) \le \liminf_{T\to\infty}[I^f(0,T,(\bar{\mathbf{x}})^-) - T\mu(f)] = 0. \tag{4.71}$$

It follows from the definition of π^f (see (4.8)) and Proposition 3.4 that

$$\pi^f(\bar{x}) = \inf\{\liminf_{T\to\infty}[I^f(0,T,v) - T\mu(f)] :\ v : [0,\infty) \to R^n$$

$$\text{is an } (f)\text{-good function such that } v(0) = \bar{x}\}. \tag{4.72}$$

Let $v : [0,\infty) \to R^n$ be an (f)-good function such that $v(0) = \bar{x}$. By (4.10), (4.11) , and Proposition 4.6, we have

$$\liminf_{T\to\infty}[I^f(0,T,v) - T\mu(f)]$$

$$= \liminf_{T\to\infty}[\Gamma^f(0,T,v) + \pi^f(v(0)) - \pi^f(v(T))] \ge \liminf_{T\to\infty}[\pi^f(\bar{x}) - \pi^f(v(T))] \ge 0.$$

When combined with (4.71) and (4.72) this implies that $\pi^f(\bar{x}) = 0$. Proposition 4.23 is proved.

Corollary 4.24. $\pi^g(\bar{x}) = 0$.

Let $x_*(t) = \bar{x}$, $t \in [0,\infty)$. By Proposition 4.5 x_* is (f)-perfect. Therefore x_* is (f)-minimal and

$$U^f(0,T,\bar{x},\bar{x}) = Tf(\bar{x},0) = T\mu(f) \text{ for all } T > 0. \tag{4.73}$$

Proof of Proposition 4.11: It is sufficient to show that there exists a unique point $x_f \in R^n$ which satisfies

$$\pi^f(x_f) \le \pi^f(z) \text{ for all } z \in R^n.$$

The function $\pi^f : R^n \to R^1$ is continuous. It follows from Proposition 7.3 and Theorem 8.1 of [41] that the function π^f possesses a point of minimum on R^n. We show its uniqueness. Assume the contrary. Then there exists a pair of points $y_1, y_2 \in R^n$ which satisfies

$$y_1 \ne y_2,\ \pi^f(y_1) = \pi^f(y_2) \le \pi^f(z) \text{ for all } z \in R^n. \tag{4.74}$$

It follows from Propositions 4.23 and 4.5 that for $i = 1, 2$, we have

$$\lim_{T\to\infty}[I^f(0,T,(\mathbf{y}_i)^+) - T\mu(f)] = \lim_{T\to\infty}[\pi^f(y_i) - \pi^f((\mathbf{y}_i)^+(T))] = \pi^f(y_i). \tag{4.75}$$

Put

$$v(t) = 2^{-1}((\mathbf{y}_1)^+(t) + (\mathbf{y}_2)^+(t)),\ t \in [0,\infty). \tag{4.76}$$

Since the function f is strictly convex it follows from (4.76) that for almost every $t \in [0,\infty)$, we have

$$f(v(t), v'(t)) \le 2^{-1}f((\mathbf{y}_1)^+(t), ((\mathbf{y}_1)^+)'(t)) + 2^{-1}f((\mathbf{y}_2)^+(t), ((\mathbf{y}_2)^+)'(t)). \tag{4.77}$$

Relation (4.74) implies that there exists a positive number δ such that for every real number $t \in [0,\delta]$ the equality $(\mathbf{y}_1)^+(t) \ne (\mathbf{y}_2)^+(t)$ is true. By this inequality, (4.76), and strict convexity of f, for almost every $t \in [0,\delta]$, we have

$$f(v(t), v'(t)) < 2^{-1}f((\mathbf{y}_1)^+(t), ((\mathbf{y}_1)^+)'(t)) + 2^{-1}f((\mathbf{y}_2)^+(t), ((\mathbf{y}_2)^+)'(t)).$$

This inequality implies that

$$\Delta := 2^{-1}I^f(0,\delta,(\mathbf{y}_1)^+) + 2^{-1}I^f(0,\delta,(\mathbf{y}_2)^+) - I^f(0,\delta,v) > 0. \tag{4.78}$$

By (4.77) and (4.78), for each real number $T > \delta$,

$$2^{-1}I^f(0,T,(\mathbf{y}_1)^+) + 2^{-1}I^f(0,T,(\mathbf{y}_2)^+) - I^f(0,T,v) \geq \Delta. \tag{4.79}$$

It follows from of (4.79), (4.8), (4.74), and (4.75) that

$$\pi^f(2^{-1}(y_1+y_2)) \leq \liminf_{T\to\infty}[I^f(0,T,v) - T\mu(f)]$$
$$= \liminf_{T\to\infty}[I^f(0,T,v) - 2^{-1}I^f(0,T,(\mathbf{y}_1)^+) - 2^{-1}I^f(0,T,(\mathbf{y}_2)^+)$$
$$+2^{-1}(I^f(0,T,(\mathbf{y}_1)^+) - \mu(f)T) + 2^{-1}(I^f(0,T,(\mathbf{y}_2)^+) - \mu(f)T)]$$
$$= \liminf_{T\to\infty}[I^f(0,T,v) - 2^{-1}I^f(0,T,(\mathbf{y}_1)^+) - 2^{-1}I^f(0,T,(\mathbf{y}_2)^+)]$$
$$+(2^{-1}\pi^f(y_1) + 2^{-1}\pi^f(y_2)) \leq \pi^f(y_1) - \Delta.$$

This contradicts (4.74). The contradiction we have reached completes the proof of Proposition 4.11.

4.6 Proofs of Theorems 4.12 and 4.13

Proof of Theorem 4.12: Let us prove Assertion 1. It is easy to see that it is sufficient to prove the following assertion:

There exists a pair of real numbers $\tau \geq 2L_1$ and $\delta_1 \in (0,1)$ such that for each real number $T \geq 2\tau_1$, each pair of points $y, z \in R^n$ satisfying $|y|, |z| \leq M$, and each a.c. function $v : [0,T] \to R^n$ which satisfies

$$v(0) = y,\ v(T) = z,\ I^f(0,T,v) \leq U^f(0,T,y,z) + \delta_1,$$

the inequality $|v(t) - \mathbf{y}^+(t)| \leq \epsilon$ is valid for all real numbers $t \in [0, L_1]$.

(Note that in order to deduce Assertion 1 we need to apply the assertion above to the function g).

Assume that this assertion does not hold. Then for each natural number $k \geq 2L_1$ there exist a real number $T_k \geq 2k$, a pair of points $y_k, z_k \in R^n$ such that

$$|y_k|, |z_k| \leq M, \tag{4.80}$$

and an a.c. function $v_k : [0, T_k] \to R^n$ which satisfies

$$v_k(0) = y_k,\ v_k(T_k) = z_k, \tag{4.81}$$

$$I^f(0,T_k,v_k) \leq U^f(0,T_k,y_k,z_k) + 1/k, \tag{4.82}$$

$$\sup\{|v_k(t) - (\mathbf{y}_k)^+(t)| :\ t \in [0, L_1]\} > \epsilon. \tag{4.83}$$

It follows from (4.80), (4.81), and Proposition 4.14 that there exists a positive number M_0 such that

$$|v_k(t)| \le M_0 \text{ for each } t \in [0, T_k] \text{ and each integer } k \ge 2L_1. \tag{4.84}$$

It follows from (4.80) and Propositions 2.6 and 4.14 that there exists a positive number M_1 such that

$$|(\mathbf{y}_k)^+(t)| \le M_1 \text{ for each } t \in [0, \infty) \text{ and each integer } k \ge 2L_1. \tag{4.85}$$

By (4.85), (4.84), A(ii), Proposition 3.8, and (4.82), for each natural number m the sequences

$$\{I^f(0, m, (\mathbf{y}_k)^+)\}_{k=2L_1}^{\infty}, \ \{I^f(0, m, v_k)\}_{k=2L_1+m}^{\infty}$$

are bounded. When combined with Proposition 3.5 this implies that there exist a strictly increasing sequence of natural numbers $\{k_j\}_{j=1}^{\infty}$ and a.c. functions $v : [0, \infty) \to R^n$ and $y : [0, \infty) \to R^n$ such that for each natural number N, we have

$$v_{k_j}(t) \to v(t) \text{ uniformly on } [0, N], \tag{4.86}$$

$$(\mathbf{y}_{k_j})^+(t) \to y(t) \text{ uniformly on } [0, N],$$

$$I^f(0, N, v) \le \liminf_{j \to \infty} I^f(0, N, v_{k_j}), \ I^f(0, N, y) \le \liminf_{j \to \infty} I^f(0, N, (\mathbf{y}_{k_j})^+). \tag{4.87}$$

By (4.86) and (4.83),

$$\sup\{|v(t) - y(t)| : \ t \in [0, L_1]\} \ge \epsilon. \tag{4.88}$$

Relations (4.86), (4.85), and (4.84) imply that

$$|v(t)| \le M_0 \text{ for all } t \in [0, \infty) \text{ and } |y(t)| \le M_1 \text{ for all } t \in [0, \infty). \tag{4.89}$$

In view of (4.88) and (4.81),

$$v(0) = y(0). \tag{4.90}$$

It follows from (4.87), (4.82), (4.86), and Proposition 3.8 that for each natural number N, we have

$$I^f(0, N, v) \le \liminf_{j \to \infty}[U^f(0, N, v_{k_j}(0), v_{k_j}(N)) + k_j^{-1}]$$

$$= \liminf_{j \to \infty}[U^f(0, N, v_{k_j}(0), v_{k_j}(N))] =$$

$$= U^f(0, N, v(0), v(N))$$

and

$$I^f(0, N, y) \le \liminf_{j \to \infty} [U^f(0, N, (\mathbf{y}_{k_j})^+(0), (\mathbf{y}_{k_j})^+(N))] = U^f(0, N, y(0), y(N)).$$

Therefore v and y are (f)-minimal functions. It follows from (4.90) and Theorems 4.9 and 4.8 that $v(t) = y(t)$ for all real numbers $t \in [0, \infty)$. This contradicts (4.88). The contradiction we have reached completes the proof of Assertion 1.

Let us prove Assertion 2. We may assume that $\epsilon < 1$. Propositions 4.14 and 2.6 imply that there exists a real number $M_0 > M$ such that the following properties hold:

(i) For each real number $T \ge 1$, each pair of points $y, z \in R^n$ which satisfies $|y|, |z| \le M$, and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$v(0) = y, \ v(T) = z, \ I^f(0, T, v) \le U^f(0, T, y, z) + 1,$$

we have

$$|v(t)| \le M_0, \ t \in [0, T].$$

(ii) For each point $y \in R^n$ which satisfies $|y| \le M$, we have

$$|\mathbf{y}^+(t)| \le M_0 \text{ and } |\mathbf{y}^-(-t)| \le M_0 \text{ for all } t \in [0, \infty).$$

It follows from the continuity of the functions π^f, π^g and Lemma 4.17 that there exists a positive number $\epsilon_0 < \epsilon$ such that for each $x_1, x_2, y_1, y_2 \in R^n$ such that

$$|x_1|, |x_2|, |y_1|, |y_2| \le M_0 + 2, \ |x_i - y_i| \le 2\epsilon_0, \ i = 1, 2 \tag{4.91}$$

and each real number $T \ge 1$, we have

$$|\pi^f(x_i) - \pi^f(y_i)| \le \epsilon/32, \ |\pi^g(x_i) - \pi^g(y_i)| \le \epsilon/32, \ i = 1, 2, \tag{4.92}$$

$$|U^f(0, T, x_1, x_2) - U^f(0, T, y_1, y_2)| \le \epsilon/32. \tag{4.93}$$

Theorem 4.7 implies that there exists a pair of real numbers $L_0 \ge 1$, $\delta_0 > 0$ such that for each real number $T \ge 2L_0$, each pair of points $y, z \in R^n$ satisfying $|y|, |z| \le M_0 + 1$, and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$v(0) = y, \ v(T) = z, \ I^f(0, T, v) \le U^f(0, T, y, z) + \delta_0,$$

we have

$$|v(t) - \bar{x}| \le \epsilon_0 \text{ for all } t \in [L_0, T - L_0]. \tag{4.94}$$

Proposition 3.8 implies that there exists a real number $\epsilon_1 \in (0, \epsilon_0)$ such that for each $x_i, y_i \in R^n$, $i = 1, 2$ which satisfy

$$|x_i|, |y_i| \le M_0, \ |x_i - y_i| \le \epsilon_1, \ i = 1, 2,$$

the following inequality holds:

$$|U^f(0, 1, x_1, x_2) - U^f(0, 1, y_1, y_2)| \le (16L_0)^{-1}\epsilon_0. \tag{4.95}$$

It follows from Assertion 1 of Theorem 4.12 that there exists an integer $L_1 \ge 2L_0$ such that for each real number $T \ge 2L_1$, each pair of points $y, z \in R^n$ satisfying $|y|, |z| \le M$, and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$v(0) = y, v(T) = z, \ I^f(0, T, v) = U^f(0, T, y, z), \tag{4.96}$$

we have

$$|v(t) - \mathbf{y}^+(t)| \le \epsilon_1 \text{ for all } t \in [0, L_0]$$

$$\text{and } |v(T - t) - \mathbf{z}^-(-t)| \le \epsilon_1 \text{ for all } t \in [0, L_0]. \tag{4.97}$$

Put

$$L_2 = 2L_1. \tag{4.98}$$

Let

$$y, z \in R^n, \ |y|, |z| \le M, \ T \ge L_2. \tag{4.99}$$

Proposition 2.16 implies that there exists an a.c. function $v : [0, T] \to R^n$ which satisfies (4.96). It follows from (4.96), (4.99), (4.98), and the choice of L_1, L_0 that the inequalities (4.97) and (4.94) are valid. By (4.96), we have

$$I^f(L_0, T - L_0, v) = U^f(0, T - 2L_0, v(L_0), v(T - L_0)). \tag{4.100}$$

In view of (4.99), (4.96), and properties (i) and (ii),

$$|v(t)| \le M_0 \text{ for all } t \in [0, T], \tag{4.101}$$

$$|\mathbf{y}^+(t)| \le M_0 \text{ and } |\mathbf{z}^-(-t)| \le M_0 \text{ for all } t \in [0, \infty).$$

It follows from (4.94), (4.101), (4.89), and the choice of ϵ_0 (see (4.93)) that

$$\epsilon/32 \ge |U^f(0, T - 2L_0, v(L_0), v(T - L_0)) - U^f(0, T - 2L_0, \bar{x}, \bar{x})|$$

$$= |U^f(0, T - 2L_0, v(L_0), v(T - L_0)) - (T - 2L_0)\mu(f)|. \tag{4.102}$$

By (4.96),

$$I^f(0, L_0, v) = U^f(0, L_0, v(0), v(L_0)),$$

$$I^f(T - L_0, T, v) = U^f(0, L_0, v(T - L_0), v(T)). \tag{4.103}$$

By (4.97), (4.101), and the choice of ϵ_0 (see (4.91)–(4.93)),

$$|U^f(0, L_0, v(0), v(L_0)) - U^f(0, L_0, \mathbf{y}^+(0), \mathbf{y}^+(L_0))| \le \epsilon/32,$$

$$|U^f(0, L_0, v(T - L_0), v(T)) - U^f(0, L_0, \mathbf{z}^-(-L_0), \mathbf{z}^-(0))| \le \epsilon/32. \quad (4.104)$$

Put

$$\tilde{z}(t) = \mathbf{z}^-(-t), \; t \in [0, \infty). \quad (4.105)$$

Since $\mathbf{y}^+$, $\mathbf{z}^-$ are (f)-minimal functions, the following equations are true:

$$U^f(0, L_0, \mathbf{y}^+(0), \mathbf{y}^+(L_0)) = I^f(0, L_0, \mathbf{y}_0^+),$$

$$U^f(0, L_0, \mathbf{z}^-(-L_0), \mathbf{z}^-(0)) = I^f(-L_0, 0, \mathbf{z}_0^-) = I^g(0, L_0, \tilde{z}). \quad (4.106)$$

Relations (4.100) and (4.102) imply that

$$|I^f(L_0, T - L_0, v) - (T - 2L_0)\mu(f)| < \epsilon/32. \quad (4.107)$$

It follows from (4.103), (4.104), and (4.106) that

$$|I^f(0, L_0, v) - I^f(0, L_0, \mathbf{y}_0^+)| < \epsilon/32. \quad (4.108)$$

In view of (4.94) and (4.97),

$$|\mathbf{y}^+(L_0) - \bar{x}| \le |\mathbf{y}^+(L_0) - v(L_0)| + |v(L_0) - \bar{x}| \le 2\epsilon_0,$$

$$|\mathbf{z}^-(-L_0) - \bar{x}| \le |\mathbf{z}^-(-L_0) - v(T - L_0)| + |v(T - L_0) - \bar{x}| \le 2\epsilon_0. \quad (4.109)$$

By (4.103), (4.104), and (4.106),

$$|I^f(T - L_0, T, v) - I^g(0, L_0, \tilde{z})| < \epsilon/32. \quad (4.110)$$

Since $\mathbf{y}^+$ is an (f)-good function it follows from (4.109) and the choice of ϵ_0 (see (4.91) and (4.92)) that

$$I^f(0, L_0, (\mathbf{y}_0)^+) = \pi^f(\mathbf{y}^+(0)) + L_0\mu(f) - \pi^f(\mathbf{y}^+(L_0))$$

$$\in [\pi^f(y) + L_0\mu(f) - \pi^f(\bar{x}) - \epsilon/32, \; \pi^f(y) + L_0\mu(f) - \pi^f(\bar{x}) + \epsilon/32]. \quad (4.111)$$

Since $\tilde{z}$ is a (g)-perfect function relations (4.109) and (4.105) and the choice of ϵ_0 (see (4.91) and (4.92)) imply that

$$I^g(0, L_0, \tilde{z}) = \pi^g(\tilde{z}(0)) + L_0\mu(f) - \pi^g(\tilde{z}(L_0))$$

$$\in [\pi^g(z) + L_0\mu(f) - \pi^g(\bar{x}) - \epsilon/32, \pi^g(z) + L_0\mu(f) - \pi^g(\bar{x}) + \epsilon/32]. \quad (4.112)$$

By (4.107), (4.108), and (4.110), we have

$$|I^f(0, T, v) - (T - 2L_0)\mu(f) - I^f(0, L_0, (\mathbf{y}_0)^+) - I^g(0, L_0, \tilde{z})| < 3\epsilon/32. \quad (4.113)$$

It follows from (4.111)–(4.113) and Corollary 4.24 that

$$(5\epsilon)/32 > |I^f(0,T,v) - (T-2L_0)\mu(f) - [\pi^f(y) + L_0\mu(f) - \pi^f(\bar{x})]$$
$$-[\pi^g(z) - L_0\mu(f) - \pi^g(\bar{x})]|$$
$$= |I^f(0,T,v) - T\mu(f) - \pi^f(y) - \pi^g(z)| = |U^f(0,T,y,z) - T\mu(f) - \pi^f(y) - \pi^g(z)|.$$

Assertion 2 is proved. This completes the proof of Theorem 4.12.

Proof of Theorem 4.13: Assertion 1 follows from Assertion 1 of Theorem 4.12 and Proposition 4.14. Let us prove Assertion 3. Proposition 4.14 implies that there exists a real number

$$M_0 > M + 1 + |\bar{x}| + |x_g| \tag{4.114}$$

such that for each real number $T \geq 1$, each point $y \in R^n$ satisfying $|y| \leq M$, and each a.c. function $v : [0,T] \to R^n$ which satisfies

$$v(0) = y, \ I^f(0,T,v) \leq \sigma^f(T,y) + 1, \tag{4.115}$$

we have

$$|v(t)| \leq M_0, \ t \in [0,T]. \tag{4.116}$$

It follows from Assertion 2 of Theorem 4.12 that there exists an integer $L_2 \geq 1$ such that for each pair of points $y, z \in R^n$ satisfying $|y|, |z| \leq M_0$ and each real number $T \geq L_2$, we have

$$|U^f(0,T,y,z) - T\mu(f) - \pi^f(y) - \pi^g(z)| \leq \epsilon. \tag{4.117}$$

Assume that a point $y \in R^n$ satisfies $|y| \leq M$ and that a real number $T \geq L_2$. In view of the choice of M_0 (see (4.114)–(4.116)), we have

$$\sigma^f(y,T) = \inf\{U^f(0,T,y,z) : \ z \in R^n\}$$
$$= \inf\{U^f(0,T,y,z) : \ z \in R^n \text{ and } |z| \leq M_0\}. \tag{4.118}$$

It follows from (4.114), (4.117), (4.118), and the choice of x^g (see Proposition 4.11) that

$$\epsilon \geq |\sigma^f(y,T) - \inf\{T\mu(f) + \pi^f(y) + \pi^g(z) : \ z \in R^n \text{ and } |z| \leq M_0\}|$$
$$= |\sigma^f(y,T) - T\mu(f) - \pi^f(y) - \pi^g(x_g)|.$$

Assertion 3 is proved.

Let us prove Assertion 2. We may assume that $\epsilon < 1$. Propositions 4.14 and 2.6 imply that there exists a real number

$$M_0 > M + |\bar{x}| + |x_g| \tag{4.119}$$

such that the following properties hold:

(iii) For each real number $T \geq 1$, each point $y \in R^n$ satisfying $|y| \leq M+1$, and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$v(0) = y, \ I^f(0, T, v) \leq \sigma^f(T, y) + 1,$$

the inequality $|v(t)| \leq M_0$ is valid for all real numbers $t \in [0, T]$.

(iv) For each point $y \in R^n$ which satisfies $|y| \leq M$ the inequalities $|\mathbf{y}^+(t)| \leq M_0$ and $|\mathbf{y}^-(-t)| \leq M_0$ are true for all real numbers $t \in [0, \infty)$.

By Theorem 4.12 there exists a pair of real numbers $\tau_0 \geq 2L_1$ and $\epsilon_0 \in (0, \min\{1, \epsilon\}/4)$ such that the following property holds:

(v) For each real number $T \geq 2\tau_0$, each point $y, z \in R^n$ satisfying $|y|, |z| \leq M_0$, and each a.c. function $v : [0, T] \to R^n$ which satisfies

$$v(0) = y, \ v(T) = z, \ I^f(0, T, v) \leq U^f(0, T, y, z) + \epsilon_0,$$

we have

$$|v(T - t) - \mathbf{z}^-(-t)| \leq \epsilon/4 \text{ for all } t \in [0, L_1]. \tag{4.120}$$

By Proposition 4.22 there exists a real number $\delta_1 > 0$ such that

$$|\mathbf{x}^-(-t) - \mathbf{y}^-(-t)| \leq \epsilon/4 \text{ for all } t \in [0, L_1] \tag{4.121}$$

for each pair of points $x, y \in R^n$ which satisfy $|x|, |y| \leq M_0+1$ and $|x-y| \leq \delta_1$.

Proposition 4.11 implies that there exists $\delta_0 \in (0, \min\{\delta_1, \epsilon_0\}/4)$ such that

$$|x_g - z| \leq \delta_1 \text{ for each } z \in R^n \text{ satisfying } |z| \leq M_0 + 1 \tag{4.122}$$

$$\text{and } |\pi^g(z) - \pi^g(x_g)| \leq 4\delta_0.$$

It follows from Assertion 2 of Theorem 4.12 and Assertion 3 of Theorem 4.13 that there exists a real number $\tau \geq \tau_0 + 1$ such that:

for each pair of point $y, z \in R^n$ satisfying $|y, |z| \leq M_0$ and each real number $T \geq \tau$, we have

$$|U^f(0, T, y, z) - T\mu(f) - \pi^f(y) - \pi^g(z)| \leq \delta_0; \tag{4.123}$$

For each point $y \in R^n$ satisfying $|y| \leq M_0$ and each real number $T \geq \tau$, we have

$$|\sigma^f(T, y) - T\mu(f) - \pi^f(y) - \pi^g(x_g)| \leq \delta_0. \tag{4.124}$$

Assume that $T \geq 2\tau$, a point $y \in R^n$ satisfies $|y| \leq M$, and that an a.c. function $v : [0, T] \to R^n$ satisfies

$$v(0) = y, \ I^f(0, T, v) \le \sigma^f(T, y) + \delta_0. \tag{4.125}$$

Put

$$z = v(T). \tag{4.126}$$

It follows from the properties (iii) and (iv), the choice of M_0 (see (4.119)), (4.125) and the inequality $|y| \le M$ that

$$|v(T)| \le M_0 \text{ for all } t \in [0, T]. \tag{4.127}$$

By (4.125) and (4.126),

$$I^f(0, T, v) \le U^f(0, T, y, z) + \delta_0. \tag{4.128}$$

By (4.125)–(4.127), relations (4.123) and (4.124) are true. Relations (4.123)–(4.126) and (4.128) imply that

$$|I^f(0, T, v) - T\mu(f) - \pi^f(y) - \pi^g(z)| \le 2\delta_0,$$

$$|I^f(0, T, v) - T\mu(f) - \pi^f(y) - \pi^g(x_g)| \le 2\delta_0.$$

By these inequalities, we have $|\pi^g(z) - \pi^g(x_g)| \le 4\delta_0$. Together with the choice of δ_0 (see (4.122)), (4.126), and (4.127) this inequality implies that

$$|x_g - z| \le \delta_1.$$

When combined with the choice of δ_1 (see (4.121)), (4.119), (4.126), and (4.127), this inequality implies that

$$|(\mathbf{x}_g)^-(-t) - \mathbf{z}^-(-t)| \le \epsilon/4 \text{ for all } t \in [0, L_1]. \tag{4.129}$$

It follows from (4.125)–(4.128) and property (v) that relation (4.120) is true. By (4.120) and (4.129), we have

$$|(\mathbf{x}_g)^-(-t) - v(T - t)| \le \epsilon/2 \text{ for all } t \in [0, L_1].$$

Assertion 2 is proved. This completes the proof of Theorem 4.13.

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